PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 2, July 1973

FAMILIES OF NEGATIVELY CURVED HERMITIAN MANIFOLDS

MICHAEL J. COWEN

ABSTRACT. A complex analytic family of compact hermitian manifolds has negative holomorphic sectional curvature in a neighborhood of any fibre having negative holomorphic sectional curvature.

1. **Introduction.** In [**1**, Hilfssatz 4, p. 120], Grauert and Reckziegel state:

If (Y, π, X) is an analytic family of compact Riemann surfaces of genus $\geqq 2$, over a Riemann surface X, then for each point x_0 in X there is a neighborhood $V \subset X$ of x_0 and a differential metric on $Y|V$ such that $Y|V$ is strongly negatively curved.

Their construction of the metric is clear, but the proof of strong negative curvature involves a computation which is not altogether complete. (The proof, however, can be completed easily using the formula for the Gaussian curvature of the sum of two hermitian metrics [**1**, Aussage 1, p. 111].) The purpose of this note is to give a simpler computation which shows that the metric actually has holomorphic sectional curvature $\leqq c < 0$ and hence *a fortiori* is strongly negatively curved [**2**, p. 39]. Indeed we will show that if the fibres of Y are n-dimensional compact manifolds each having holomorphic sectional curvative less than a negative constant, then $Y|V$ has negative holomorphic sectional curvature.

2. **Definitions and statement of results.** Let ds^2 be a hermitian metric on an n-dimensional complex manifold M, with $ds^2 = \sum g_{ij}\, dz_i\, d\bar{z}_j$ in local coordinates. Define the curvature tensor by

$$K_{ijkm} = \frac{\partial^2 g_{ij}}{\partial z_k\, \partial \bar{z}_m} - \sum_{p,q} \frac{\partial g_{ip}}{\partial z_k} g^{pq} \frac{\partial g_{qj}}{\partial \bar{z}_m} \quad \text{for } 1 \leqq i, j, k, m \leqq n.$$

M has holomorphic sectional curvature (which will be denoted by h.s.c.) less than a constant c if $-\sum k_{ijkm} s_i \bar{s}_j s_k \bar{s}_m < c$ for all holomorphic unit tangent vectors $s = \sum s_i\, \partial/\partial z_i$.

Received by the editors June 5, 1972.

AMS (MOS) subject classifications (1970). Primary 53C55; Secondary 32G05, 32H20.

Key words and phrases. Holomorphic sectional curvature, hyperbolic manifold, analytic family, differential metric.

THEOREM. *Let (Y, π, X) be an analytic family of compact complex n-dimensional manifolds over a Riemann surface X, such that for each $x_0 \in X$, the fibre $Y_{x_0}=\pi^{-1}(x_0)$ has a hermitian metric of h.s.c. $<c<0$, then there exists a neighborhood V of x_0 in X such that $Y|V$ has a hermitian metric of h.s.c. $<c'<0$, with c, c' constants.*

REMARK. This generalizes Grauert and Reckziegel's result since a compact Riemann surface of genus $\geqq 2$ has a hermitian metric of Gaussian curvature (equals h.s.c. on a Riemann surface) less than a negative constant [**2**, Theorem 5.1, p. 12].

COROLLARY. *If σ is a holomorphic section of Y with isolated singularities in X, then σ extends as a holomorphic section to all of X.*

3. **Construction of the metric.** The construction is the obvious generalization of that in [**1**].

Since (Y, π, X) is locally trivial we can find a neighborhood V with coordinate z_{n+1} centered at x_0, and neighborhoods $U_1, \cdots, U_r$ in Y such that $Y|V=\bigcup U_m$, each U_m has coordinates $z_1, \cdots, z_{n+1}$ with

$$\pi(z_1, \cdots, z_{n+1}) = z_{n+1},$$

and $z_1, \cdots, z_n$ give coordinates in $U_m \cap Y_{z_{n+1}}$ for all z_{n+1} in V. The hermitian metric on $Y_0=Y_{x_0}$ is of the form $\sum g_{ij}(z_1, \cdots, z_n)\, dz_i\, d\bar{z}_j$ on $U_m \cap Y_0$ and thus can be extended to a pseudo-hermitian metric $\sum h_{ij}\, dz_i\, d\bar{z}_j$ on U_m, $1 \leqq i, j \leqq n$, by setting $h_{ij}(z_1, \cdots, z_{n+1})=g_{ij}(z_1, \cdots, z_n)$. These pseudo-hermitian metrics can then be patched together by a partition of unity to give a pseudo-hermitian metric α on $Y|V$, such that $\alpha|Y_0$ is the original hermitian metric on Y_0. That is, $\alpha=\sum k_{ij}\, dz_i\, d\bar{z}_j$ $(1 \leqq i, j \leqq n+1)$ on U_m and $k_{ij}(z_1, \cdots, z_n, 0)=g_{ij}(z_1, \cdots, z_n)$ for $1 \leqq i, j \leqq n$. Since $\alpha|Y_0$ has h.s.c. $<c<0$, it is clear that for z_{n+1} close enough to 0, $\alpha|Y_{z_{n+1}}$ will have h.s.c. $<c<0$. By shrinking V we can assume (1) $V=\{|z_{n+1}|<t\}$, (2) $\alpha|Y_{z_{n+1}}$ has h.s.c. $<c<0$ for all $z_{n+1} \in V$, and (3) k_{ij}, its first, and second partial derivatives are bounded on each U_m for $1 \leqq i, j \leqq n+1$. Since a disc in $\boldsymbol{C}$ has a hermitian metric of Gaussian curvature -1, we can put V in a larger disc and obtain a metric $h(z_{n+1})\, dz_{n+1}\, d\bar{z}_{n+1}$ on V of Gaussian curvature $=-1$ such that h, its first, and second partials are bounded on V. Define a metric ds^2 on $Y|V$ for each $\lambda>0$ by $ds^2=\alpha+\lambda h(z_{n+1})\, dz_{n+1}\, d\bar{z}_{n+1}$, i.e. on U_m, $ds^2=\sum k_{ij}\, dz_i\, d\bar{z}_j+\lambda h(z_{n+1})\, dz_{n+1}\, d\bar{z}_{n+1}$. Note that for large λ, ds^2 has negative h.s.c. in both the fibre and base directions. We wish to choose λ_0 so that for all $\lambda \geqq \lambda_0$, ds^2 will have h.s.c. $\leqq c''<0$. Clearly it suffices to do this on each U_m and then take the maximum of the λ_0's so obtained.

4. **Proof of negative sectional curvature.** Assume we have shown the following:

(i) $K_{ijkm} \to \tilde{K}_{ijkm}$ for $1 \leqq i, j, k, m \leqq n$, uniformly on U_m as $\lambda \to \infty$, where $\tilde{K}_{ijkm}(z_1, \cdots, z_{n+1})$ is the curvature of ds^2 restricted to $Y_{z_{n+1}} \cap U_m$.

(ii) $|K_{ijkm}| \leqq M$ on U_m for all $1 \leqq i, j, k, m \leqq n+1$ except when $i=j=k=m=n+1$, and M is a constant.

(iii)
$$K_{ijkm} = \lambda\left(\frac{\partial^2 h}{\partial z_{n+1}\,\partial \bar{z}_{n+1}} - \frac{1}{h}\frac{\partial h}{\partial z_{n+1}}\frac{\partial h}{\partial \bar{z}_{n+1}}\right) + O(1),$$

when $i=j=k=m=n+1$, where $O(1)$ means a term which is uniformly bounded on U_m.

Since the Gaussian curvature of h is

$$\frac{1}{h}\left(\frac{\partial^2 h}{\partial z_{n+1}\,\partial \bar{z}_{n+1}} - \frac{1}{h}\frac{\partial h}{\partial z_{n+1}}\frac{\partial h}{\partial \bar{z}_{n+1}}\right) \leqq -1$$

and h is bounded on U_m, we have:

(iii)′ $K_{ijkm} \leqq \lambda c'$ when $i=j=k=m=n+1$, where $c'<0$ is a constant, for $\lambda \geqq \lambda_0$.

Fix $z=(z_1, \cdots, z_{n+1})$. If $s=\sum_{i=1}^n s_i(\partial/\partial z_i)$ is a holomorphic unit tangent vector to the fibre $Y_{z_{n+1}}$ then by (i) we have

$$-\sum K_{ijkm}s_i\bar{s}_j s_k \bar{s}_m \to -\sum \tilde{K}_{ijkm}s_i\bar{s}_j s_k\bar{s}_m < c < 0 \quad \text{as } \lambda \to \infty.$$

Hence by compactness of the unit sphere, we can choose λ_0 large enough so that for $\lambda \geqq \lambda_0$ we have $-\sum K_{ijkm}s_i\bar{s}_j s_k\bar{s}_m < c$ for s tangent to the fibre. But if $s=\sum_{i=1}^{n+1} s_i\,\partial/\partial z_i$ is any holomorphic unit tangent vector, then by (ii) and (iii)′ we have:

$$(*)\qquad \begin{aligned} -\sum K_{ijkm}s_i\bar{s}_j s_k\bar{s}_m &\leqq -\sum_{i,j,k,m=1}^{n} K_{ijkm}s_i\bar{s}_j s_k\bar{s}_m \\ &\quad + M\sum{}' |s_i|\,|s_j|\,|s_k|\,|s_m| + \lambda c'\,|s_{n+1}|^4, \end{aligned}$$

where $\sum'$ is the sum of the terms where at least one, but not all, of the i, j, k, m equals $n+1$. Thus if s is not tangent to the fibre, i.e., $s_{n+1} \neq 0$, then by taking λ_0 large enough we can insure that the h.s.c. is less than c_s in a neighborhood of s on the unit sphere, for all $\lambda \geqq \lambda_0$. But from $(*)$ it is also clear that if s is tangent to the fibre, then the h.s.c. is less than c_s in a neighborhood of s for all $\lambda \geqq \lambda_0$. Therefore for each fixed z the h.s.c. at z is less than c_z for $\lambda > \lambda_0$ and hence by the relative compactness of U_m, the h.s.c. $<c<0$ on U_m for $\lambda \geqq \lambda_0$, which proves the Theorem.

Let $ds^2|\,Y_{z_{n+1}} = \sum \tilde{k}_{ij}\,dz_i\,d\bar{z}_j$ be the metric restricted to the fibre, where $\tilde{k}_{ij}=k_{ij}$ for $1 \leqq i, j \leqq n$. Since

$$ds^2 = \sum k_{ij}\,dz_i\,d\bar{z}_j + \lambda h(z_{n+1})\,dz_{n+1}\,d\bar{z}_{n+1} \equiv \sum g_{ij}\,dz_i\,d\bar{z}_j$$

where $1 \leqq i, j \leqq n+1$, it is easy to check that:

(a) $g^{ij} = \tilde{k}^{ij} + O(\lambda^{-1})$, $\partial g_{ip}/\partial z_k = \partial \tilde{k}_{ip}/\partial z_k$, $\partial^2 g_{ij}/\partial z_k \, \partial \bar{z}_m = \partial^2 \tilde{k}_{ij}/\partial z_k \, \partial \bar{z}_m$ for $1 \leqq i, j, k, m, p \leqq n$.

(b)
$$g^{ij} = \lambda^{-1} h(z_{n+1})^{-1} + O(\lambda^{-2}), \qquad \frac{\partial g_{ip}}{\partial z_{n+1}} = O(1) + \frac{\partial h}{\partial z_{n+1}},$$

$$\frac{\partial^2 g_{ij}}{\partial z_{n+1} \, \partial \bar{z}_{n+1}} = O(1) + \lambda \frac{\partial^2 h}{\partial z_{n+1} \, \partial \bar{z}_{n+1}} \quad \text{for } i = j = p = n + 1.$$

(c) $g^{ij} = O(\lambda^{-1})$, $\partial g_{ip}/\partial z_k = O(1)$, $\partial^2 g_{ij}/\partial z_k \, \partial \bar{z}_m = O(1)$ otherwise. (*Note.* Since h is a function only of z_{n+1}, terms such as $\partial g_{ip}/\partial z_k$, for $i = p = n+1$ but $k \neq n+1$, do not involve λ or the derivatives of h.)

If $1 \leqq i, j, k, m \leqq n$ then

$$\begin{aligned} K_{ijkm} &= \frac{\partial^2 \tilde{k}_{ij}}{\partial z_k \, \partial \bar{z}_m} - \sum_{p,q=1}^{n} \frac{\partial \tilde{k}_{ip}}{\partial z_k} (\tilde{k}^{pq} + O(\lambda^{-1})) \frac{\partial \tilde{k}_{qj}}{\partial \bar{z}_k} + O(\lambda^{-1}) \\ &= \tilde{K}_{ijkm} + O(\lambda^{-1}), \end{aligned}$$

which proves (i). If $i = j = k = m = n+1$, then

$$\begin{aligned} K_{ijkm} &= \lambda \frac{\partial^2 h}{\partial z_{n+1} \, \partial \bar{z}_{n+1}} + O(1) - \sum_{p,q=1}^{n} \frac{\partial g_{ip}}{\partial z_k} O(\lambda^{-1}) \frac{\partial g_{qj}}{\partial \bar{z}_m} \\ &\quad - \sum_{p=1}^{n} \frac{\partial g_{ip}}{\partial z_k} O(\lambda^{-1}) O(1) - \sum_{q=1}^{n} O(1) O(\lambda^{-1}) \frac{\partial g_{qj}}{\partial \bar{z}_k} \\ &\quad - (O(1) + \lambda \, \partial h/\partial z_{n+1})(\lambda^{-1} h^{-1} + O(\lambda^{-2}))(O(1) + \lambda(\partial h/\partial \bar{z}_{n+1})) \\ &= \lambda(\partial^2 h/\partial z_{n+1} \, \partial \bar{z}_{n+1}) - h^{-1}(\partial h/\partial z_{n+1})(\partial h/\partial \bar{z}_{n+1}) \\ &\quad + O(1) + O(\lambda^{-1}) + O(\lambda^{-2}) \end{aligned}$$

which proves (iii). The proof of (ii) is obvious, since the only terms which are not $O(1)$ or $O(\lambda^{-1})$ are those appearing only when $i = j = k = m = n+1$.

5. **Proof of Corollary.** Assume σ has an isolated singularity at $x_0 \in H$. By the Theorem, there is a neighborhood $V = \{|z| < 1\}$ of x_0 such that $Y|V$ has a metric of h.s.c. $< c < 0$. Thus by [**2**, Theorem 4.11, p. 61], $Y|V$ is hyperbolic and, by a theorem of Mrs. Kwack [**2**, Theorem 3.1, p. 83], $\sigma: V - \{0\} \to Y|V$ has a holomorphic extension to $\sigma': V \to Y|V$ if there exists a suitable sequence of points $x_n \to x_0$ such that $\sigma(x_n) \to p_0 \in Y|V$. Since $Y|V$ is relatively compact in Y, the result follows.

6. **Remarks.** That X is a Riemann surface was not crucial to the proof of the Theorem and the proof goes through with obvious modifications when X is an arbitrary complex manifold. Then in the Corollary, σ

need only have singularities contained in an analytic set of codimension $\geqq 1$ in X, for σ to extend to all of X. The proof of the Corollary then follows from a result of Mrs. Kwack [**2**, Theorem 4.1, p. 86].

References

1. H. Grauert and H. Reckziegel, *Hermitesche Metriken und normale Familien holomorpher Abbildungen*, Math. Z. **89** (1965), 108–125. MR **33** #2827.

2. S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, Pure and Appl. Math., no. 2, Marcel Dekker, New York, 1970. MR **43** #3503.

Department of Mathematics, Tulane University, New Orleans, Louisiana 70118

Current address: Department of Mathematics, Princeton University, Princeton, New Jersey 08540