FAMILIES OF NEGATIVELY CURVED HERMITIAN MANIFOLDS
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Abstract. A complex analytic family of compact hermitian manifolds has negative holomorphic sectional curvature in a neighborhood of any fibre having negative holomorphic sectional curvature.

1. Introduction. In [1, Hilfssatz 4, p. 120], Grauert and Reckziegel state:

If \((Y, \pi, X)\) is an analytic family of compact Riemann surfaces of genus \(\geq 2\), over a Riemann surface \(X\), then for each point \(x_0\) in \(X\) there is a neighborhood \(V \subset X\) of \(x_0\) and a differential metric on \(Y|V\) such that \(Y|V\) is strongly negatively curved.

Their construction of the metric is clear, but the proof of strong negative curvature involves a computation which is not altogether complete. (The proof, however, can be completed easily using the formula for the Gaussian curvature of the sum of two hermitian metrics [1, Aussage 1, p. 111].) The purpose of this note is to give a simpler computation which shows that the metric actually has holomorphic sectional curvature \(\leq c < 0\) and hence a fortiori is strongly negatively curved [2, p. 39]. Indeed we will show that if the fibres of \(Y\) are \(n\)-dimensional compact manifolds each having holomorphic sectional curvature less than a negative constant, then \(Y|V\) has negative holomorphic sectional curvature.

2. Definitions and statement of results. Let \(ds^2\) be a hermitian metric on an \(n\)-dimensional complex manifold \(M\), with \(ds^2 = \sum g_{ij} \, dz_i \, d\bar{z}_j\) in local coordinates. Define the curvature tensor by

\[
K_{ijkm} = \frac{\partial^2 g_{ij}}{\partial z_k \partial \bar{z}_m} - \sum_{p,q} g_{ip} \frac{\partial g_{kj}}{\partial z_q} \frac{\partial g_{mq}}{\partial \bar{z}_j} \quad \text{for} \quad 1 \leq i, j, k, m \leq n.
\]

\(M\) has holomorphic sectional curvature (which will be denoted by h.s.c.) less than a constant \(c\) if \(- \sum k_{ijkm} s_i s_j s_k s_m < c\) for all holomorphic unit tangent vectors \(s = \sum s_i \partial / \partial z_i\).

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Theorem. Let \((Y, \pi, X)\) be an analytic family of compact complex \(n\)-dimensional manifolds over a Riemann surface \(X\), such that for each \(x_0 \in X\), the fibre \(Y_{x_0} = \pi^{-1}(x_0)\) has a hermitian metric of h.s.c. \(<c<0\), then there exists a neighborhood \(V\) of \(x_0\) in \(X\) such that \(Y|V\) has a hermitian metric of h.s.c. \(<c'<0\), with \(c, c'\) constants.

Remark. This generalizes Grauert and Reckziegel's result since a compact Riemann surface of genus \(\geq 2\) has a hermitian metric of Gaussian curvature (equals h.s.c. on a Riemann surface) less than a negative constant [2, Theorem 5.1, p. 12].

Corollary. If \(\sigma\) is a holomorphic section of \(Y\) with isolated singularities in \(X\), then \(\sigma\) extends as a holomorphic section to all of \(X\).

3. Construction of the metric. The construction is the obvious generalization of that in [1].

Since \((Y, \pi, X)\) is locally trivial we can find a neighborhood \(V\) with coordinate \(z_{n+1}\) centered at \(x_0\), and neighborhoods \(U_1, \ldots, U_r\) in \(Y\) such that \(Y|V = \bigcup U_m\), each \(U_m\) has coordinates \(z_1, \ldots, z_{n+1}\) with

\[
\pi(z_1, \ldots, z_{n+1}) = z_{n+1},
\]

and \(z_1, \ldots, z_n\) give coordinates in \(U_m \cap Y_{z_{n+1}}\) for all \(z_{n+1}\) in \(V\). The hermitian metric on \(Y_0 = Y_{x_0}\) is of the form \(\sum g_{ij}(z_1, \ldots, z_n) \, dz_i \, d\bar{z}_j\) on \(U_m \cap Y_0\) and thus can be extended to a pseudo-hermitian metric \(\sum h_{ij} \, dz_i \, d\bar{z}_j\) on \(U_m\), \(1 \leq i, j \leq n\), by setting \(h_{ij}(z_1, \ldots, z_{n+1}) = g_{ij}(z_1, \ldots, z_n)\). These pseudo-hermitian metrics can then be patched together by a partition of unity to give a pseudo-hermitian metric \(\alpha\) on \(Y|V\), such that \(\alpha|Y_0\) is the original hermitian metric on \(Y_0\). That is, \(\alpha = \sum k_{ij} \, dz_i \, d\bar{z}_j\) \((1 \leq i, j \leq n+1)\) on \(U_m\) and \(k_{ij}(z_1, \ldots, z_n, 0) = g_{ij}(z_1, \ldots, z_n)\) for \(1 \leq i, j \leq n\). Since \(\alpha|Y_0\) has h.s.c. \(<c<0\), it is clear that for \(z_{n+1}\) close enough to 0, \(\alpha|Y_{z_{n+1}}\) will have h.s.c. \(<c<0\). By shrinking \(V\) we can assume (1) \(V = (|z_{n+1}| < t)\), (2) \(\alpha|Y_{z_{n+1}}\) has h.s.c. \(<c<0\) for all \(z_{n+1} \in V\), and (3) \(k_{ij}\), its first, and second partial derivatives are bounded on each \(U_m\) for \(1 \leq i, j \leq n+1\). Since a disc in \(C\) has a hermitian metric of Gaussian curvature \(-1\), we can put \(V\) in a larger disc and obtain a metric \(h(z_{n+1}) \, dz_{n+1} \, d\bar{z}_{n+1}\) on \(V\) of Gaussian curvature \(-1\) such that \(h\), its first, and second partials are bounded on \(V\).

Define a metric \(ds^2\) on \(Y|V\) for each \(\lambda > 0\) by \(ds^2 = \alpha + \lambda h(z_{n+1}) \, dz_{n+1} \, d\bar{z}_{n+1}\), i.e. on \(U_m\), \(ds^2 = \sum k_{ij} \, dz_i \, d\bar{z}_j + \lambda h(z_{n+1}) \, dz_{n+1} \, d\bar{z}_{n+1}\). Note that for large \(\lambda\), \(ds^2\) has negative h.s.c. in both the fibre and base directions. We wish to choose \(\lambda_0\) so that for all \(\lambda \geq \lambda_0\), \(ds^2\) will have h.s.c. \(\leq c' < 0\). Clearly it suffices to do this on each \(U_m\) and then take the maximum of the \(\lambda_0\)'s so obtained.
4. Proof of negative sectional curvature. Assume we have shown the following:

(i) $K_{ijklm} \to \tilde{K}_{ijklm}$ for $1 \leq i, j, k, m \leq n$, uniformly on $U_m$ as $\lambda \to \infty$, where $\tilde{K}_{ijklm}(z_1, \ldots, z_{n+1})$ is the curvature of $ds^2$ restricted to $Y_{n+1} \cap U_m$.

(ii) $|K_{ijklm}| \leq M$ on $U_m$ for all $1 \leq i, j, k, m \leq n+1$ except when $i=j=k=m=n+1$, and $M$ is a constant.

(iii) $K_{ijklm} = \lambda \left( \frac{\partial^2 h}{\partial z_{n+1} \partial z_{n+1}} - \frac{1}{h} \frac{\partial h}{\partial z_{n+1}} \frac{\partial h}{\partial z_{n+1}} \right) + O(1)$, 

when $i=j=k=m=n+1$, where $O(1)$ means a term which is uniformly bounded on $U_m$.

Since the Gaussian curvature of $h$ is

$$\frac{1}{h} \left( \frac{\partial^2 h}{\partial z_{n+1} \partial z_{n+1}} - \frac{1}{h} \frac{\partial h}{\partial z_{n+1}} \frac{\partial h}{\partial z_{n+1}} \right) \leq -1$$

and $h$ is bounded on $U_m$, we have:

(iii)' $K_{ijklm} \leq \lambda c'$ when $i=j=k=m=n+1$, where $c'<0$ is a constant, for $\lambda \geq \lambda_0$.

Fix $z=(z_1, \ldots, z_{n+1})$. If $s=\sum_{i=1}^n s_i(\partial/\partial z_i)$ is a holomorphic unit tangent vector to the fibre $Y_{n+1}$ then by (i) we have

$$-\sum K_{ijklm} s_i s_j s_k s_m \to -\sum \tilde{K}_{ijklm} s_i s_j s_k s_m < 0 \text{ as } \lambda \to \infty.$$ 

Hence by compactness of the unit sphere, we can choose $\lambda_0$ large enough so that for $\lambda \geq \lambda_0$ we have $-\sum K_{ijklm} s_i s_j s_k s_m < 0$ for $s$ tangent to the fibre. But if $s=\sum_{i=1}^{n+1} s_i \partial/\partial z_i$ is any holomorphic unit tangent vector, then by (ii) and (iii)' we have:

$$-\sum K_{ijklm} s_i s_j s_k s_m \leq -\sum_{i,j,k,m=1}^n K_{ijklm} s_i s_j s_k s_m + M \sum |s_i| |s_j| |s_k| |s_m| + \lambda c' |s_{n+1}|^4,$$

where $\sum$ is the sum of the terms where at least one, but not all, of the $i, j, k, m$ equals $n+1$. Thus if $s$ is not tangent to the fibre, i.e., $s_{n+1}\neq 0$, then by taking $\lambda_0$ large enough we can insure that the h.s.c. is less than $c_s$ in a neighborhood of $s$ on the unit sphere, for all $\lambda \geq \lambda_0$. But from (*) it is also clear that if $s$ is tangent to the fibre, then the h.s.c. is less than $c_s$ in a neighborhood of $s$ for all $\lambda \geq \lambda_0$. Therefore for each fixed $z$ the h.s.c. at $z$ is less than $c_s$ for $\lambda > \lambda_0$ and hence by the relative compactness of $U_m$, the h.s.c. $c<0$ on $U_m$ for $\lambda \geq \lambda_0$, which proves the Theorem.

Let $ds^2|Y_{n+1} = \sum k_{ij} dz_i d\bar{z}_j$ be the metric restricted to the fibre, where $k_{ij}=k_{ji}$ for $1 \leq i, j \leq n$. Since

$$ds^2 = \sum k_{ij} dz_i d\bar{z}_j + \lambda h(z_{n+1}) dz_{n+1} d\bar{z}_{n+1} \equiv \sum g_{ij} dz_i d\bar{z}_j$$

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where $1 \leq i, j \leq n+1$, it is easy to check that:

(a) \( g^{ij} = \lambda^{-1} h(z_{n+1})^{-1} + O(\lambda^{-2}) \),

\( \frac{\partial g_{ip}}{\partial z_{n+1}} = O(1) + \frac{\partial h}{\partial z_{n+1}} \),

\( \frac{\partial^2 g_{ij}}{\partial z_{n+1} \partial z_{n+1}} = O(1) + \frac{\lambda}{\partial^2 z_{n+1} \partial z_{n+1}} \) for \( i = j = p = n + 1 \).

(b) \( g^{ij} = k^{ij} + O(1) + \frac{\partial h}{\partial z_{n+1}} \).

(c) \( g^{ij} = O(\lambda^{-1}) \),

\( \frac{\partial h}{\partial z_{n+1}} = O(1) \) otherwise. (Note. Since \( h \) is a function only of \( z_{n+1} \), terms such as \( \frac{\partial g_{ip}}{\partial z_k} \), for \( i = p = n + 1 \) but \( k \neq n + 1 \), do not involve \( \lambda \) or the derivatives of \( h \).)

If \( 1 \leq i, j, k, m, p \leq n \) then

\[
K_{ijkm} = \frac{\partial^2 k_{ij}}{\partial z_k \partial z_m} - \sum_{p=1}^{n} \frac{\partial k_{ip}}{\partial z_k} (k^{pq} + O(\lambda^{-1})) \frac{\partial k_{qj}}{\partial z_k} + O(\lambda^{-1})
\]

which proves (i). If \( i = j = k = m = n + 1 \), then

\[
K_{ijkm} = \lambda \frac{\partial^2 h}{\partial z_{n+1} \partial z_{n+1}} + O(1) - \sum_{p=1}^{n} \frac{\partial g_{ip}}{\partial z_k} O(\lambda^{-1}) \frac{\partial g_{qj}}{\partial z_m} - \sum_{p=1}^{n} \frac{\partial g_{ip}}{\partial z_k} O(\lambda^{-1}) O(1) \frac{\partial g_{qj}}{\partial z_m} - (O(1) + \lambda \frac{\partial h}{\partial z_{n+1}})(\lambda^{-1} h^{-1} + O(\lambda^{-2})) (O(1) + \lambda (\frac{\partial h}{\partial z_{n+1}}))
\]

which proves (iii). The proof of (ii) is obvious, since the only terms which are not \( O(1) \) or \( O(\lambda^{-1}) \) are those appearing only when \( i = j = k = m = n + 1 \).

5. Proof of Corollary. Assume \( \sigma \) has an isolated singularity at \( x_0 \in H \).

By the Theorem, there is a neighborhood \( V = \{ |z| < 1 \} \) of \( x_0 \) such that \( Y \mid V \) has a metric of h.s.c. \( c < 0 \). Thus by [2, Theorem 4.11, p. 61], \( Y \mid V \) is hyperbolic and, by a theorem of Mrs. Kwack [2, Theorem 3.1, p. 83], \( \sigma: V \to Y \mid V \) has a holomorphic extension to \( \sigma': V \to Y \mid V \) if there exists a suitable sequence of points \( x_n \to x_0 \) such that \( \sigma(x_n) \to p_0 \in Y \mid V \). Since \( Y \mid V \) is relatively compact in \( Y \), the result follows.

6. Remarks. That \( X \) is a Riemann surface was not crucial to the proof of the Theorem and the proof goes through with obvious modifications when \( X \) is an arbitrary complex manifold. Then in the Corollary, \( \sigma \)
need only have singularities contained in an analytic set of codimension \( \geq 1 \) in \( X \), for \( \sigma \) to extend to all of \( X \). The proof of the Corollary then follows from a result of Mrs. Kwack [2, Theorem 4.1, p. 86].

References


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