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REGULAR CLOSED MAPS

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ABSTRACT. A subset A of X is far from the remainder if whenever $\mathcal{U}$ is a free open ultrafilter in X there exists $U \in \mathcal{U}$ such that $A \cap \mathrm{cl}_X U = \varnothing$. A map is regular closed provided that the image of every regular closed set is closed. In this note we use some recent results of G. Viglino to show that every map can be extended to a regular closed map with far from the remainder point inverses. We also relate these maps to several other interesting classes of maps.

Introduction. Throughout this paper all spaces are assumed to be Hausdorff and all functions are assumed to be continuous. A subset A of a space X is a *regular closed* subset of X if $A = \mathrm{cl}_X(\mathrm{int}\, A)$ and a map $f: X \to Y$ is a *regular closed map* if for every regular closed set $A \subset X, f(A)$ is closed in Y. A subset A of X is said to be *far from the remainder* (f.f.r.) if for every free open ultrafilter $\mathcal{U}$ in X, there exists $V \in \mathcal{U}$ such that $\mathrm{cl}_X V \cap A = \varnothing$.

An *extension* of a map $f: X \to Y$ is a map $F: Z \to Y$ such that X is a proper dense subset of Z and $F|X = f$ and we say that a space X is *f-absolutely closed* (alternately f is absolutely closed) provided that no extension of $f: X \to Y$ exists. In [4] G. Viglino gave some characterizations of absolutely closed maps. In this note we give another such characterization and relate these maps to several interesting classes of maps.

A characterization.

THEOREM 1 [4, THEOREM 1.2]. *A map $f: X \to Y$ is absolutely closed provided that whenever $\mathcal{U}$ is an open filter in X such that $f(\mathcal{U})$ converges in Y, $\mathcal{U}$ has a nonempty adherent set in X.*

THEOREM 2. *A map $f: X \to Y$ is absolutely closed if and only if it is regular closed and point inverses are f.f.r.*

PROOF OF THE NECESSITY. Let A be a regular closed subset of X and suppose that $y \in \mathrm{cl}_Y f(A) \backslash f(A)$. Let $\mathcal{N}(y)$ denote the open neighborhood filter of y. Then for every $N \in \mathcal{N}(y), f^{-1}(N) \cap A$ is nonempty as is $f^{-1}(N) \cap \mathrm{int}\, A$. Thus $\{f^{-1}(N) \cap \mathrm{int}\, A : N \in \mathcal{N}(y)\}$ is a base for an open filter $\mathcal{U}$ such

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that $f(\mathscr{U})$ converges to y in Y. By Theorem 1, $\mathrm{ad}_X\mathscr{U}$, the adherent set of $\mathscr{U}$ in X, is a nonempty subset of $f^{-1}(y)$. But if $x \in \mathrm{ad}_X\mathscr{U}$, x belongs to A and so $f(x)=y \in f(A)$. This contradiction implies that f is regular closed.

In order to see that point inverses are f.f.r. let $y \in Y$ and let $\mathscr{U}$ be a free open ultrafilter. Then by Theorem 1 and the maximality of $\mathscr{U}$, $y \notin \mathrm{ad}_Y f(\mathscr{U})$. Thus there exists an open set V containing y and $U \in \mathscr{U}$ such that $V \cap f(U)=\varnothing$. Then $f^{-1}(V) \cap U=\varnothing$ and $f^{-1}(y) \cap \mathrm{cl}_X U=\varnothing$. Thus $f^{-1}(y)$ is f.f.r.

THE SUFFICIENCY. Suppose that $\mathscr{U}$ is a free open filter in X and suppose that $f(\mathscr{U})$ converges to $y \in Y$. Since X is a regular closed set, $f(X)$ is a closed subset of Y so that $y \in f(X)$. By our hypothesis $f^{-1}(y)$ is f.f.r., hence there exists $U \in \mathscr{U}$ such that $f^{-1}(y) \cap \mathrm{cl}_X U=\varnothing$. But then $A=\mathrm{cl}_X U$ is regular closed and $f(A)$ is a closed set not containing y. This is impossible since $Y \backslash f(A)$ would then be a neighborhood of y containing no $V \in \mathscr{U}$. This contradiction implies that f is absolutely closed.

COROLLARY 2.1. *Every map can be extended to a regular closed map with f.f.r. point inverses.*

This follows immediately from Theorem 2 and Theorem (2.2) of [**4**].

REMARKS. The following question is posed in [**4**]: Is every closed map with point inverses H-closed relative to X an absolutely closed map? We have seen that replacing the term "closed map" in the question with the weaker term "regular closed map", and strengthening the condition "point inverses are H-closed relative to X" to "point inverses are f.f.r." gives a characterization of absolutely closed maps. The question in [**4**] may then be rephrased as follows: If point inverses of a closed map are H-closed relative to X, are they also f.f.r.? One can easily show that if point inverses of a map are f.f.r., they are also H-closed relative to X. There are H-closed subsets which are not f.f.r. See [**1**, p. 48].

Let τX denote the Katětov H-closed extension of X [3]. A map $f: X \to Y$ is called *τ-proper* provided there exists a map $\tau f: \tau X \to \tau Y$ such that $\tau f|_X = f$ and a τ-proper map is said to be *τ-perfect* if $\tau f(\tau X \backslash X) \subset (\tau Y \backslash Y)$. In [**1**, Theorem 5] A. Blaszczyk and J. Mioduszewski showed that every τ-perfect map is absolutely closed. Every open map is τ-proper [**1**] so τ-proper maps are not necessarily absolutely closed. The example of [**1**, p. 49] shows that absolutely closed maps are not necessarily τ-proper. By Theorem 2 and Theorem 3 of [**1**], a τ-proper map is absolutely closed if and only if it is τ-perfect.

THEOREM 3. *Every τ-proper map $f: X \to Y$ can be extended to a τ-perfect map.*

PROOF. Let $Z=(\tau f)^{-1}(Y)$ and $g=\tau f|Z$. Then τZ is an absolute closure of X and by [**3**, Theorem (1.13)] there exists a map $h: \tau X \to \tau Z$ such that

$h|X$ is the identity. Also τX is an absolute closure of Z so there exists a map $g:\tau Z\to\tau X$ such that $g|Z$ is the identity. Thus h and g are inverses and τX is the Katětov extension of Z. Since g can be extended to τX, and $g(\tau X\setminus X)\subset(\tau Y\setminus Y)$, g is τ-perfect as required.

REMARKS. The author has been able to show that every map can be extended in a unique manner to a perfect map, however the extension often behaves badly in that the domain usually fails to be a Hausdorff space [**2**]. One would expect this to be the case since a nonperfect map on an absolutely closed space cannot be extended to a perfect map on a Hausdorff space. Since every map on an absolutely closed space is regular closed and every set is f.f.r. we avoid this difficulty in the above extension.

REFFRENCES

1. A. Blaszczyk and J. Mioduszewski, *On factorization of maps through* τX, Colloq. Math. **23** (1971), 45–52.

2. R. F. Dickman, Jr., *On closed extensions of functions*, Proc. Nat. Acad. Sci. U.S.A. **62** (1969), 326–332. MR **42** #5233.

3. C. T. Liu, *Absolutely closed spaces*, Trans. Amer. Math. Soc. **130** (1968), 86–104.

4. G. Viglino, *Extensions of functions and spaces*, Trans. Amer. Math. Soc. (to appear).

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