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# INDECOMPOSABLE CONTINUA IN STONE-ČECH COMPACTIFICATIONS

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ABSTRACT. We show that if $Y$ is a continuum irreducible from $a$ to $b$, which is connected im Kleinen and first countable at $b$, and if $X=Y-\{b\}$, then $\beta X-X$ is an indecomposable continuum. Examples are given showing that both first countability and connectedness im Kleinen are needed here. We also show that $\beta[0, 1)-[0, 1)$ has a strong near-homogeneity property.

1. **Introduction.** In [2] and [3] it is shown that if $X=[0, 1)$ then $\beta X-X$ is an indecomposable continuum; here $\beta X$ is the Stone-Čech compactification of $X$. In [7], Dickman showed that among locally connected spaces, $[0, 1)$ is essentially the only such space. In this paper we exhibit other types of spaces $X$ with this property. We shall also show that for $X=[0, 1)$, $\beta X-X$ is stably almost homogeneous, a concept to be defined below.

The set function $T$ has been studied and applied in [1], [5], [6], [8], [9], [11], and [14]. We follow these papers in writing $T(p)$ for $T(\{p\})$. This set function will be used in the argument at one point and familiarity with it is assumed. Familiarity with [10], [12], [13], and [15] is also assumed. If we write $X=A\cup B$ sep, then we mean that $\mathrm{Cl}(A)\cap B=\varnothing$ and $A\cap \mathrm{Cl}(B)=\varnothing$ while neither $A$ nor $B$ is empty. By $f: X\simeq Y$, we mean $f$ is a homeomorphism of $X$ onto $Y$.

2. **Indecomposable continua in $\beta X$.**

LEMMA 1. *There is a covariant functor $\beta$ on the category of Tychonoff spaces and continuous maps such that for any Tychonoff space $X$, $\beta X$ is the Stone-Čech compactification of $X$ and if $f: X\to Y$ then $\beta f:\beta X\to\beta Y$ is the unique extension of $f$ induced by $f$ treated as a map from $X$ to $\beta Y$.*

*Notation.* If $X$ is a Tychonoff space, let $\gamma X=\beta X-X$. If $f$ is a continuous map from $X$ to $Y$, let $\gamma f$ denote $\beta f|\gamma X$.

DEFINITION. Let $Y$ be a compact Hausdorff continuum irreducible from $a$ to $b$ such that $Y$ is both connected im Kleinen and first countable

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at $b$. Let $X=Y-\{b\}$. Then we call the topological pair $(Y, X)$ a *wave* from $a$ to $b$.

By stringing together indecomposable continua, a wave $(Y, X)$ can be constructed such that $Y$ is not connected im Kleinen at any point of $X$.

LEMMA 2. *If $Y$ is a compact Hausdorff continuum irreducible from $a$ to $b$ and $x \in Y$, $T(x)$ either separates $a$ from $b$, contains $a$, or contains $b$. In case $T(x)$ separates $a$ from $b$, $Y-T(x)$ has exactly two components, $A$ and $B$, where $a \in A$ and $b \in B$, and both $T(x) \cup A$ and $T(x) \cup B$ are proper subcontinua of $Y$ containing $a$ and $b$ respectively.*

REMARK ON PROOF. This lemma can be established using standard techniques and Theorem 1.10 of [**14**], since each $x \in Y$ different from $a$ and $b$ weakly separates $a$ from $b$.

LEMMA 3. *If $Y$ is a compact Hausdorff continuum irreducible from $a$ to $b$ and $W \subseteq Y$ is a continuum with $b \in \operatorname{Int} W$, then $W-\{b\}$ is connected.*

PROOF. If $W-\{b\}=M_0 \cup N_0$ sep, let $M=M_0 \cup \{b\}$; $N=N_0 \cup \{b\}$. Then $b$ lies in the boundary of $M$ and $N$ and, by Theorem 6 of [**15**, p. 194], each of $M$ and $N$ is nowhere dense, so that $M \cup N=W$ is nowhere dense also, a contradiction.

LEMMA 4. *If $(Y, X)$ is a wave from $a$ to $b$, and $Z$ is a Hausdorff compactification of $X$, then $Z-X$ is a Hausdorff continuum.*

PROOF. Since $Y$ is connected im Kleinen and first countable at $b$, there exists a denumerable collection of continua $\{N_i\}_{i=1}^{\infty}$ such that for each $i$, $b \in \operatorname{Int}(N_i)$ and $N_{i+1} \subseteq N_i$ and $\bigcap_{i=1}^{\infty} N_i=\{b\}$. It is readily seen that

$$Z - X = \operatorname{Cl}_Z(N_i - \{b\}) - N_i = \bigcap_{i=1}^{\infty} \operatorname{Cl}_Z(N_i - \{b\}).$$

Then, since each $N_i-\{b\}$ is connected by Lemma 3, $Z-X$ is an intersection of a monotone collection of continua.

LEMMA 5. *Let $X$ be a compact Hausdorff space, $b \in X$, $\{b\}$ a component of $X$, and suppose $X$ is first countable at $b$ and $\langle b_i \rangle_{i=1}^{\infty}$ is a sequence in $X-\{b\}$ converging to $b$. Then there exist two closed subsets $A$ and $B$ of $X$ such that $A \cup B=X$, $A \cap B=\{b\}$, and each of $A$ and $B$ contains infinitely many (that is, a subsequence) of the $b_i$'s.*

PROOF. It is readily seen that there exists a neighborhood basis $\{N_j\}_{j=1}^{\infty}$ at $b$ consisting of closed and open sets such that $N_{j+1} \subseteq N_j$ for each $j$ and $N_1=X$; by passing to a subset if necessary, we may suppose that each

$N_j - N_{j+1}$ contains at least one of the $b_i$'s. Then set

$$A = \{b\} \cup \bigcup_{j=1}^{\infty} (N_{2j-1} - N_{2j}), \qquad B = \{b\} \cup \bigcup_{j=1}^{\infty} (N_{2j} - N_{2j+1}).$$

Then $A$ and $B$ have the desired properties.

LEMMA 6. *If $(Y, X)$ is a wave from $a$ to $b$ and $W$ is a nondegenerate subcontinuum of $Y$ containing $b$, then $b \in \operatorname{Int} W$.*

PROOF. Suppose not. Then let $p \in W$, $p \neq b$. Since $b \notin T(p)$, by connectedness im Kleinen at $b$, it follows that either $a \in T(p)$ or $Y - T(p) = A \cup B$ sep, where $a \in A$ and $b \in B$. If $a \in T(p)$, $T(p) \cup W$ is a proper subcontinuum of $Y$ containing both $a$ and $b$; if $a \notin T(p)$, $A \cup T(p) \cup W$ is such a continuum, and in either case we have a contradiction.

COROLLARY 1. *If $(Y, X)$ is a wave from $a$ to $b$ and $M$ is a closed subset of $Y$ with $b \in M$ but $b \notin \operatorname{Int} M$, $\{b\}$ is a component of $M$.*

LEMMA 7. *If $Y$ is a compact Hausdorff space first countable at a point $b$, then $Y - \{b\}$ is normal.*

PROOF. Let $\{O_k\}_{k=1}^{\infty}$ be a countable basis of open neighborhoods at $b$. Then $Y - \{b\} = \bigcup_{k=1}^{\infty} (Y - O_k)$, so that $Y - \{b\}$ is sigma compact and hence Lindelöf. Then $Y - \{b\}$ is paracompact [**10**, p. 174, 6.5] and hence normal [**10**, p. 163, 2.2].

THEOREM 1. *If $(Y, X)$ is a wave from $a$ to $b$, then $\gamma X$ is an indecomposable continuum.*

PROOF. By Lemma 4, $\gamma X$ is a continuum. Suppose $F$ is a proper subcontinuum of $\gamma X$ which contains an interior point $q$ with respect to $\gamma X$. Let $p \in \gamma(X) - F$. Let $U$ and $V$ be open sets in $\beta X$ with $\operatorname{Cl}(U) \cap \operatorname{Cl}(V) = \operatorname{Cl}(U) \cap (\gamma X - \operatorname{Int} F) = \operatorname{Cl}(V) \cap F = \varnothing$ while $p \in V$ and $q \in U$. This is possible by regularity.

Then $X \cap V$ and $X \cap U$ are open subsets of $X$ and hence of $Y$ since $X$ is open in $Y$. Let $\langle b_i \rangle_{i=1}^{\infty}$ be a sequence of points in $U \cap X$ converging in $Y$ to $b$. This is possible since $b \in \operatorname{Cl}_Y(U \cap X)$ and $Y$ is first countable at $b$.

Then $\{b\}$ is a component of $Y - (V \cap X)$, by Corollary 1, and $\langle b_i \rangle$ is a sequence in $(Y - V) - \{b\}$ converging to $b$. By Lemma 6 there are two closed sets $A_0$ and $B_0$ such that $A_0 \cup B_0 = Y - V$, $A_0 \cap B_0 = \{b\}$, and each of $A_0$ and $B_0$ contains a subsequence of the $b_i$'s. Let $A = A_0 \cap X$, $B = B_0 \cap X$. Then $A$ and $B$ are disjoint closed subsets of $X$, and since $X$ is normal, disjoint closed sets lie in disjoint zero sets, and by Theorem 6.5 III of [**12**], $\operatorname{Cl}_{\beta X}(A) \cap \operatorname{Cl}_{\beta X}(B) = \varnothing$. Now since each of $A$ and $B$ contains infinitely many of the $b_i$'s, it follows that each of $\operatorname{Cl}_{\beta X}(A)$ and $\operatorname{Cl}_{\beta X}(B)$ contains points of $\mathrm{Cl}_{\beta X}(U)\cap\gamma X$, and hence points of $F$. Thus, since if $x\in\gamma X-\mathrm{Cl}_{\beta X}(A\cup B)$, it follows that $x\in\mathrm{Cl}_{\beta X}(V)$ and hence $x\notin F$, we have $F=(F\cap\mathrm{Cl}_{\beta X}(A))\cup(F\cap\mathrm{Cl}_{\beta X}(B))$ sep, so that $F$ is no continuum.

COROLLARY 2 ([**2**] AND [**3**]). *Let $X=[0,1)$. Then $\gamma X$ is an indecomposable continuum.*

EXAMPLE 1. Let $L$ denote the long line, consisting of $\omega_1\times[0,1)$ with the lexicographic order, where $\omega_1$ is the first uncountable ordinal; we take the order topology on $L$. Then consider $L\times[0,1]$ with the product topology. Let

$$X=\{((\alpha,t),s)\in L\times[0,1]:t=0 \text{ or } t=s\}.$$

Let $Y=X\cup\{b\}$ be the one-point compactification of $X$. Then $Y$ is irreducible from $((0,0),1)$ to $b$ and is connected im Kleinen at $b$. $(Y,X)$ fails to be a wave from $a$ to $b$ because $Y$ is not first countable at $b$.

Standard techniques applied to continuous functions from $\omega_1$ to $[0,1]$ yield the result that $\gamma X\cong[0,1]$ in this case. Thus, first countability cannot be dispensed with in the hypothesis of Theorem 1. Connectedness im Kleinen also cannot be removed from the hypothesis of Theorem 1; the usual topologist's sin $1/x$ curve, with $b$ taken from the limit arc, yields a decomposable continuum as $\gamma X$.

LEMMA 8. *If $X$ is a Tychonoff space and $Z$ is any compactification of $X$ with inclusion map $i:X\to Z$, then $\gamma i(\gamma X)=Z-i(X)$.*

REMARK ON PROOF. This is a special case of Theorem 6.12 of [**12**, p. 92].

LEMMA 9. *If $X$ and $Y$ are Tychonoff spaces and $f:X\cong Y$, then $\gamma f$: $\gamma X\cong\gamma Y$.*

PROOF. By Lemma 8, $\gamma f(\gamma X)=\gamma Y$ and since $\beta$ is a functor it follows that $\beta f$ is a homeomorphism since it has inverse $\beta(f^{-1})$. Then $\gamma f$ is a homeomorphism since it is a restriction of one.

LEMMA 10. *Let $X$ be a normal Hausdorff space and $A$ a closed subset of $X$ such that $X-A$ contains a closed but not compact subset of $X$. Then $\gamma X-\mathrm{Cl}_{\beta X}(A)$ is a nonempty, open subset of $\gamma X$.*

LEMMA 11. *Suppose $X$ is a Tychonoff space and $f:X\cong X$ is the identity inside some closed subset $V$ of $X$. Then $\gamma f:\gamma X\cong\gamma X$ is the identity inside $\gamma X\cap\mathrm{Cl}_{\beta X}(V)$.*

DEFINITION. We say a topological space $X$ is *almost homogeneous* if for any $p,q\in X$, and any neighborhood $U$ of $q$ there is a homeomorphism $h:X\cong X$ such that $h(p)\in U$. If, in addition, we may choose $h$ to be the identity on some nonempty open subset of $X$, we say $X$ is *stably almost homogeneous.*

THEOREM 2. *Let* $X=[0, 1)$; *then* $\gamma X$ *is a stably almost homogeneous indecomposable continuum.*

PROOF. Throughout this proof, Cl denotes $\mathrm{Cl}_{\beta X}$. Let $x, y \in \gamma X$ and let $V_0$ be any open set in $\gamma X$ containing $y$. Then $V_0=V_1\cap\gamma X$ for some $V_1$ open in $\beta X$. Then there exists a $V_2$ open in $\beta X$ such that $y\in V_2\subseteq \mathrm{Cl}\, V_2\subseteq V_1$ and $x\notin \mathrm{Cl}\, V_2$ unless $x=y$, in which case there is nothing to prove. Let $U_0$ be open in $\beta X$ with $x\in U_0$ and $\mathrm{Cl}\, U_0\cap\mathrm{Cl}\, V_2=\varnothing$. Now let $U=U_0\cap X$ and $V=V_2\cap X$. We shall assume, with no loss of generality, that $0<\inf U<\inf V$.

Now, define four sequences $\langle p_n\rangle_{n=1}^{\infty}$, $\langle q_n\rangle_{n=1}^{\infty}$, $\langle r_n\rangle_{n=1}^{\infty}$, and $\langle s_n\rangle_{n=1}^{\infty}$ as follows: $p_1=\inf U$. Whenever $p_i$ has been defined, set $q_i=\sup\{t\in U: [p_i, t]\cap V=\varnothing\}$. When $q_i$ has been defined, set $r_i=\inf\{t\in V: t>q_i\}$. When $r_i$ has been defined, set $s_i=\sup\{t\in V: [r_i, t]\cap U=\varnothing\}$. When $s_i$ has been defined, set $p_{i+1}=\inf\{t\in U: t>s_i\}$. This completes the recursive definition of the four sequences. They have the following properties: $p_i<q_i<r_i<s_i<p_{i+1}$ for each $i$; the limit in $[0, 1]$ of each sequence is 1, $U\subseteq\bigcup_{i=1}^{\infty}[p_i, q_i]$, and $V\subseteq\bigcup_{i=1}^{\infty}[r_i, s_i]$. We now choose two more sequences $\langle x_i\rangle_{i=1}^{\infty}$ and $\langle y_i\rangle_{i=1}^{\infty}$ so that, for each $i$, $r_i<x_i<y_i<s_i$ and the closed interval $[x_i, y_i]$ is a subset of $V$. Finally we choose two more sequences $\langle a_i\rangle_{i=1}^{\infty}$ and $\langle b_i\rangle_{i=1}^{\infty}$ such that $a_1=0$; $0<b_1<p_1$, and for $i>1$ we choose $s_i<a_{i+1}<b_{i+1}<p_{i+1}$. Now define $h: X\cong X$ as follows: For each $i$,

(1) $h$ is the identity on $[a_i, b_i]$,
(2) $h$ maps the interval $[b_i, p_i]$ linearly onto $[b_i, x_i]$,
(3) $h$ maps $[p_i, q_i]$ linearly onto $[x_i, y_i]$,
(4) $h$ maps $[q_i, a_{i+1}]$ linearly onto $[y_i, a_{i+1}]$.

Then $h(U)\subseteq V$, and hence $\beta h(\mathrm{Cl}(U))\subseteq\mathrm{Cl}(V)$, and since $x\in\mathrm{Cl}(U)$, $\beta h(x)\in\mathrm{Cl}(V)\subseteq\mathrm{Cl}(V_2)\subseteq V_1$, and $\beta h(x)=\gamma h(x)\in V_0$ as desired. Furthermore, $\gamma h$ is the identity inside the set $\gamma X\cap\mathrm{Cl}(\bigcup_{i=1}^{\infty}[a_i, b_i])$, which contains a nonvoid open subset of $\gamma X$ by Lemma 10, setting the closed set $\bigcup_{i=1}^{\infty}[b_i, a_{i+1}]$ equal to $A$ in the lemma.

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