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PERIPHERALLY COMPACT TREE-LIKE SPACES ARE CONTINUUM-WISE CONNECTED

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ABSTRACT. The theorem stated in the title is proved, and examples are given to show the need for both peripheral compactness and tree-likeness.

Two recent papers by Proizvolov ([2], [3]) provide much interesting and useful information about the structure of tree-like spaces. Among other facts, Proizvolov proves [2] that the density of a peripherally compact tree-like space equals its weight; each tree-like peripherally compact space X has a unique tree-like compactification which must have the same weight as that of X and [3] any two disjoint closed subsets of a peripherally compact tree-like space can be separated by a closed set; tree-like peripherally compact spaces are hereditarily normal and dyadic tree-like compact spaces are metrizable. Basic to the proofs of all these theorems is the theorem of the title. A paper by Gurin [1] is the reference cited, but it is not easy to see how to derive this assertion from his results. This note is devoted to an elementary proof of the assertion and some relevant examples.

A topological space is said to be tree-like if it is connected, Hausdorff and every two points are separated by a third point. A peripherally compact space is a space with an open basis whose elements have compact boundaries. It is easy to see that a peripherally compact tree-like space has an open basis consisting of sets with finite boundaries. Therefore, each peripherally compact tree-like space has an open base consisting of connected sets with finite boundaries (see [4, p. 19]).

PROOF OF THE THEOREM. Let X be a peripherally compact tree-like space and let p and q be two points of X. Define M to be the subset of X consisting of p, q and the points which separate p from q. Arguments given by Whyburn [4, pp. 42, 43 and 51] show that M is compact and

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naturally ordered by the cut point order from p to q. (The arguments given by Whyburn are only claimed to work for separable metric space but they work equally well in our situation.) If M is not connected, it is the union of disjoint closed sets H and K such that H contains p and K contains q. Because M is compact and naturally ordered, H has a largest element h and K has a smallest element k which succeeds h. But then any point which separates h and k is not in M but separates p from q, a contradiction. Therefore M is a continuum joining p and q.

EXAMPLE 1. A tree-like space which is not peripherally compact, not continuum-wise connected and not locally connected. The space is the subspace of the plane $A \cup B \cup C$, where $A = \{(x, y) : x = 0, 0 < y \leqq 1\}$, $B = \{(x, y) : 0 \leqq x \leqq 1$ and y is the reciprocal of a positive integer$\}$, and $C = \{(\frac{1}{2}, 0)\}$.

EXAMPLE 2. A peripherally compact connected space which is not tree-like and not continuum-wise connected. The space is the subspace of the plane $A \cup B$ where $A = \{(0, y) : y$ is a number$\}$ and

$$B = \{(x, (1/x \operatorname{Sin}(1/x)) : 0 < x \leqq 1\}.$$

This space is, in fact, locally compact.

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