

SOME RELATIONS BETWEEN SEMIGROUPS OF POLYHEDRA¹

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ABSTRACT. All spaces are CW-complexes of finite type. The notation “=” means homotopy equivalence. The following theorems are proved:

- (1) If $X \vee A = Y \vee A$ then there exists T such that $\Omega X \times T = \Omega Y \times T$,
- (2) $X \times A = Y \times A$ implies that there is a T such that $\Sigma X \vee T = \Sigma Y \vee T$.

A partial converse is also proved. As a corollary we get that if $X \vee A = Y \vee A$ then $\pi_*(X) = \pi_*(Y)$ and if $X \times A = Y \times A$ then $h_*(X) = h_*(Y)$ for many homology theories.

1. Introduction. We work with the set W of based homotopy equivalence classes of connected CW-complexes of finite type, i.e., complexes with finitely many cells in each dimension. We are deliberately going to confuse a space with its equivalence class. We may also want to consider the subset W_1 of all 1-connected complexes.

We will denote the one point union by $X + Y$.

We have two semigroup structures on this set, namely $X + Y$ and $X \times Y$; these can be used to define two equivalence classes and Grothendieck groups: $G_1 = G_1(W)$, $G_2 = G_2(W)$, by setting $X \sim_1 Y$ if and only if there exists an A in W such that $X + A = Y + A$, and $X \sim_2 Y$ if and only if there exists an A such that $X \times A = Y \times A$.

We have chosen to work with complexes of finite type because of what we need for Corollaries 2.11 and 2.12. If we want to pass to a larger class of spaces, some care has to be taken, because for any X and Y , $X + A = Y + A$ if we set $A = (\bigoplus_{i=1}^{\infty} X) + (\bigoplus_{i=1}^{\infty} Y)$. Similarly $X \sim_2 Y$.

These equivalence relations have been investigated by a number of people; among them: Freyd ([2], [3]), Hilton ([4], [5]), Hilton-Roitberg [7]. An exposition of Freyd's work can be found in Cohen [1].

Freyd worked in the stable range and proved many interesting results there about G_1 . Among other things he completely determined the structure

Received by the editors September 24, 1972.

AMS (MOS) subject classifications (1970). Primary 55D10, 55D35, 55D40, 55E20.

Key words and phrases. Noncancellation, suspension, one point union, loop spaces, product, homotopy groups, homology groups.

¹ This work is part of the author's doctoral dissertation, written at Cornell University under Professor P. J. Hilton.

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of G_1 for stable compact polyhedra, and gave the first example of two spaces X, Y such that $X \neq Y$, but $X \sim_1 Y$ stably. His example can also serve as an example in the nonstable range.

Hilton gives some nonstable examples of the same phenomenon, i.e., $X \not\sim_1 Y$ and $X \neq Y$. He also proves Corollary 2.11 where X, Y, A are assumed to be suspensions, and Corollary 2.12 for ordinary homology.

Hilton-Roitberg considered the second equivalence class and gave a few examples of $X \sim_2 Y$ with $X \neq Y$. They are especially interested in finite-dimensional examples. They produced examples with $X \times S^3 = Y \times S^3$.

As a further nonsimply connected example we prove

PROPOSITION. $L_{p,q} \sim_1 L_{p,\bar{q}}$ where $L_{p,q}$ denotes a three-dimensional lens space.

PROOF. It is well known that $L_{p,q} = P_p \cup_\alpha e^3$ where P_p is the pseudo projective plane with fundamental group Z_p and α is some map. Similarly, $L_{p,\bar{q}} = P_p \cup_\beta e'^3$. Moreover, $\pi_2 L_{p,\bar{q}} = \pi_2 L_{p,q} = 0$. So

$$L_{p,q} \vee S^3 = P_p \cup_\alpha e^3 \vee S^3 = P_p \cup_\alpha e^3 \cup_\beta e'^3 = L_{p,\bar{q}} \vee S^3.$$

It appears certain that the two equivalence classes are different, and Freyd's [2] examples should serve as an example where $X \sim_1 Y$ but $X \not\sim_2 Y$.

Here we are concerned with relations between the two equivalence relations. More specifically, if we denote by W_1 the subset of W consisting of all 1-connected spaces, then Theorem 2.4 could be formulated as follows: The map $\Omega: W_1 \rightarrow W$ which sends $X \rightarrow \Omega X$ can be extended to a map (not a homomorphism) $\Omega: G_1(W_1) \rightarrow G_2(W)$ of the Grothendieck groups. Dually, the map $\Sigma: W \rightarrow W_1$ sending X to ΣX can be extended to a map $\Sigma: G_2(W) \rightarrow G_1(W_1)$.

If we denote by $H' \subset G_1(W_1)$ the subset of all equivalence classes, each of which contains an H' -space, and $H \subset G_2(W)$ the subset of equivalence classes containing H -spaces, then Theorems 2.10 and 2.7 state that $\Omega|_{H'}: H' \rightarrow G_2(W)$ is injective, and that $\Sigma|_H: H \rightarrow G_1(W_1)$ is injective.

From Theorems 2.1 and 2.4 we can easily see that if $X \sim_1 Y$ in W_1 or $X \sim_2 Y$ in W , then they have the same homology and homotopy groups.

In fact, more is true. If $X \sim_1 Y$ or $X \sim_2 Y$, it can be easily shown that $\Gamma_i(X) \cong \Gamma_i(Y)$, where Γ is the Whitehead gamma group [8], that the isomorphisms $\pi_i(X) \cong \pi_i(Y)$, $H_i(X) \cong H_i(Y)$ can be chosen to commute with the Hurewicz homomorphism, and that for finite dimensional spaces $H^*(X)$ and $H^*(Y)$ are isomorphic both as modules over the Steenrod algebra and as rings.

I wish to thank Professor P. Hilton for his advice and encouragement.

2. **Proofs of the theorems.** In what follows we will make essential use of the following identities for homotopy types.

$$(\alpha) \quad \Sigma(X \times Y) = \Sigma X + \Sigma Y + \Sigma(X \wedge Y) \quad [6, \text{p. 104}],$$

$$(\beta) \quad \Omega(X + Y) = \Omega X \times \Omega Y \times \Omega \Sigma(\Omega X \wedge \Omega Y) \quad [6, \text{p. 216}].$$

THEOREM 2.1. *If $X \times A = Y \times A$ and X, Y, A are in W , then there exists $T \in W_1$ which is also a suspension so that $\Sigma X + T = \Sigma Y + T$.*

PROOF. Let $\alpha_1 = \Sigma X, \beta_1 = \Sigma Y, \alpha_2 = \Sigma A + \Sigma(X \wedge A), \beta_2 = \Sigma A + \Sigma(Y \wedge A)$. Then, using (α) and the fact that $\Sigma(X \times A) = \Sigma(Y \times A)$, we get that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$. Define $\alpha_n = \Sigma A^n + \Sigma(X \wedge A^n), \beta_n = \Sigma A^n + \Sigma(Y \wedge A^n)$, where A^n denotes the n -fold smash product. Then we have that:

$$\begin{aligned} \alpha_n + \alpha_{n+1} &= \Sigma A^n + \Sigma(X \wedge A^n) + \Sigma A^{n+1} + \Sigma(X \wedge A^{n+1}) \\ &= \Sigma A^n + A^n \wedge (\Sigma X + \Sigma A + \Sigma(X \wedge A)). \end{aligned}$$

Using (α) again, we get that the last term is equal to $\beta_n + \beta_{n+1}$. Let $T = \bigoplus_{i=2}^{\infty} \alpha_i$. Then $T = \bigoplus_{i=2}^{\infty} \beta_i$, because

$$\bigoplus_{i=2}^{\infty} \alpha_i = \bigoplus_{i=1}^{\infty} (\alpha_{2i} + \alpha_{2i+1}) = \bigoplus_{i=2}^{\infty} (\beta_{2i} + \beta_{2i+1}).$$

Moreover

$$\begin{aligned} \Sigma X + T &= \bigoplus_{i=1}^{\infty} \alpha_i = \bigoplus_{i=1}^{\infty} (\alpha_{2i-1} + \alpha_{2i}) = \bigoplus_{i=1}^{\infty} (\beta_{2i-1} + \beta_{2i}) \\ &= \bigoplus_{i=1}^{\infty} \beta_i = \Sigma Y + T. \end{aligned}$$

The proofs of the other theorems are similar, though more involved; we need some preliminaries first:

DEFINITION. Let F be the minimal set of functions; $f: W \times W \rightarrow W$ so that:

- (a) The function $P_1 \in F$ where $P_1(X, Y) = X$.
The function $f(X) = *$ is in F .
- (b) The function $P_2 \in F$ where $P_2(X, Y) = Y$.
The function $P_3 \in F$ where $P_3(X, Y) = C$ for some fixed $C \in W$.
- (c) If g, f are in F then $f \times g \in F$ where $(f \times g)(X, Y) = f(X, Y) \times g(X, Y)$.
- (d) If f_1, \dots, f_n are in F , then $\Omega \Sigma(f_1 \wedge \dots \wedge f_n) \in F$.

So every function in F can be obtained from P_1, P_2 , and P_3 by a sequence of operations $\alpha_1 \dots \alpha_n$ of type (c) or (d). We will call such a sequence a presentation of f . All our constructions in F will actually be constructions on elements (f, α) , where α is a presentation of f .

DEFINITION. For any $f \in F$, $\text{deg } f$ will be defined as follows:

- (a) $\text{deg } P_1=0, \text{deg } * = \infty,$
- (b) $\text{deg } P_2=0, \text{deg } P_3=0,$
- (c) $\text{deg } f \times g = \min\{\text{deg } f, \text{deg } g\},$
- (d) $\text{deg } \Omega\Sigma(f_1 \wedge \dots \wedge f_n) = \text{deg } f_1 + \dots + \text{deg } f_n + (n-1).$

It is obvious from the definition that, for any $X, Y \in W$ and $f \in F, \text{deg } f \leq$ connectivity of $f(X, Y).$

LEMMA 2.2. Let $g, f_1 \dots f_n \in F.$ Then

$$\begin{aligned} \Omega\Sigma(f_1 \wedge \dots \wedge (f_n \times g)) \\ = \Omega\Sigma(f_1 \wedge \dots \wedge f_n) \times \Omega\Sigma(f_1 \wedge \dots \wedge f_{n-1} \wedge g) \times \alpha \end{aligned}$$

where $\alpha \in F$ and

$$\text{deg } \alpha > \max\{\text{deg } \Omega\Sigma(f_1 \wedge \dots \wedge f_n), \text{deg } \Omega\Sigma(f_1 \wedge \dots \wedge f_{n-1} \wedge g)\}.$$

PROOF. To prove this we use the identities (α) and $(\beta).$ Write f for $f_1 \wedge \dots \wedge f_{n-1}.$ Then

$$\begin{aligned} \Omega\Sigma(f \wedge (f_n \times g)) &= \Omega(\Sigma f \wedge f_n + \Sigma f \wedge g + \Sigma f \wedge f_n \wedge g) \\ &= \Omega\Sigma(f \wedge f_n) \times \Omega\Sigma(f \wedge g) \times \Omega\Sigma(f \wedge f_n \wedge g) \\ &\quad \times \Omega\Sigma((\Omega\Sigma f \wedge f_n) \wedge (\Omega\Sigma f \wedge g)) \\ &\quad \times \Omega\Sigma([\Omega\Sigma(f \wedge f_n) \times \Omega\Sigma(f \wedge g) \times \Omega\Sigma((\Omega\Sigma f \wedge f_n) \wedge (\Omega\Sigma f \wedge g))] \\ &\quad \wedge \Omega\Sigma(f \wedge f_n \wedge g)). \end{aligned}$$

Hence α is defined by the previous equation, and obviously has the right properties.

THEOREM 2.3. Let P, Q, B be in $W,$ and $\varphi, \varphi \in F,$ with $\text{deg } \varphi > 0,$ such that $P \times B \times \varphi(P, B) = Q \times B \times \varphi(Q, B).$ Then there exists $T \in W$ such that $\varphi(P, B) \times T = \varphi(Q, B) \times T.$

As an immediate corollary we have:

THEOREM 2.4. Let X, Y, A be in W_1 satisfying $X + A = Y + A.$ Then there exists $T \in W$ such that $\Omega X \times T = \Omega Y \times T.$

PROOF. Use (β) to get

$$\Omega X \times \Omega A \times \Omega\Sigma(\Omega X \wedge \Omega A) = \Omega Y \times \Omega A \times \Omega\Sigma(\Omega Y \wedge \Omega A).$$

The theorem will follow from Theorem 2.3 by putting $P = \Omega X, Q = \Omega Y, B = \Omega A, \varphi = \Omega\Sigma(P_1 \wedge P_2),$ and $\psi = P_1.$

DEFINITION AND LEMMA 2.5. For any $f \in F$ we will define $\tilde{f} \in F.$ If P, Q, B satisfy the assumptions of Theorem 2.3, we will have $f(P, B) \times \tilde{f}(P, B) = f(Q, B) \times \tilde{f}(Q, B).$ Moreover, $\text{deg } f \leq \text{deg } \tilde{f}.$

PROOF. The definition and the proof will be by induction on the length of the presentation of f :

(a) $\bar{P}_1 = P_2 \times \varphi$, $\bar{*} = *$. Both claims about f follow directly from the assumptions of Theorem 2.3.

(b) $\bar{P}_2 = *$, $\bar{P}_3 = *$.

(c) Suppose \bar{f}_1 and \bar{f}_2 are defined. Define $\overline{f_1 \times f_2} = \bar{f}_1 \times \bar{f}_2$. Again, the verification is immediate.

(d) Suppose $\bar{f}_1, \dots, \bar{f}_n$ are defined. Let $f = \Omega\Sigma(f_1 \wedge \dots \wedge f_n)$. Then by Lemma 2.2 and induction on n , we get

$$\Omega\Sigma[(f_1 \times \bar{f}_1) \wedge \dots \wedge (f_n \times \bar{f}_n)] = \left(\prod \Omega\Sigma(f_1^{\varepsilon_1} \wedge \dots \wedge f_n^{\varepsilon_n})\right) \times \beta$$

where $\varepsilon_i = 0, 1$ and by definition $f^0 = f$, $f^1 = \bar{f}$, the product runs over all possible combinations, and $\beta \in F$ with $\deg \beta > \deg f$. Define $\bar{f} = \left(\prod \Omega\Sigma(f_1^{\varepsilon_1} \dots f_n^{\varepsilon_n})\right) \times \beta$ where the product runs over all possible combinations with $\varepsilon_i = 1$ for at least one i . The statement about the degrees is immediate, and the other statement follows from the fact that

$$\begin{aligned} &(\Omega\Sigma[(f_1 \times \bar{f}_1) \wedge \dots \wedge (f_n \times \bar{f}_n)])(P, B) \\ &= (\Omega\Sigma[(f_1 \times \bar{f}_1) \wedge \dots \wedge (f_n \times \bar{f}_n)])(Q, B). \end{aligned}$$

For any $f \in F$ define $f^0 = f$ and $f^n = \overline{f^{n-1}}$. Then:

LEMMA 2.6. *For any $f \in F$ so that $f \neq *$, there exists $n > 0$ such that $\deg f^n > \deg f$.*

PROOF. The proof is again by induction:

(a) We have that $P_1^2 = \bar{\varphi}$. Hence $n = 2$ since $\deg \varphi \geq 1$.

(b) As $\bar{P}_2 = *$ and $\bar{P}_3 = *$, we have that $n = 1$.

(c) If $\deg f^n > \deg f$ and $\deg g^m > \deg g$, then $\deg(f \times g)^{\max(n,m)} > \deg(f \times g)$.

(d) Suppose the assertion is true for f_1, \dots, f_n , i.e., there exists m such that $\deg f_i^m > \deg f_i$ for $i \leq n$. Let $f = \Omega\Sigma(f_1 \wedge \dots \wedge f_n)$. By induction on k one gets that $f^k = \prod \Omega\Sigma(f_1^{\varepsilon_1} \wedge \dots \wedge f_n^{\varepsilon_n}) \times \gamma$, where all the elements in the product satisfy $\varepsilon_1 + \dots + \varepsilon_n \geq k$, and $\gamma \in F$ satisfies $\deg \gamma > \deg f$. If we take $k = nm$, then each element $\Omega\Sigma(f_1^{\varepsilon_1} \wedge \dots \wedge f_n^{\varepsilon_n})$ in the product must have at least one ε_i satisfying $\varepsilon_i \geq m$. That will give us that $\deg f^k > \deg f$.

PROOF OF THEOREM 2.3. Consider the infinite product $\prod_{i \geq 0} \psi^i(P, B)$. The product is in W because $\deg \psi^i \rightarrow \infty$ by Lemma 2.6. Then by Lemma 2.5:

$$\begin{aligned} \prod_{i \geq 0} \psi^i(P, B) &= \prod_{i \geq 0} (\psi^{2i} \times \psi^{2i+1})(P, B) \\ &= \prod_{i \geq 0} (\psi^{2i} \times \psi^{2i+1})(Q, B) = \prod_{i \geq 0} \psi^i(Q, B). \end{aligned}$$

Moreover,

$$\begin{aligned} \prod_{i \geq 1} \psi^i(P, B) &= \left(\prod_{i \geq 1} \psi^{2i-1} \times \psi^{2i} \right) (P, B) \\ &= \prod_{i \geq 1} (\psi^{2i-1} \times \psi^{2i})(Q, B) = \prod_{i \geq 1} \psi^i(Q, B). \end{aligned}$$

Putting $T = \prod_{i \geq 1} \psi^i(P, B)$ we get the result.

We will turn now to a partial converse of Theorem 2.1.

THEOREM 2.7. *If X, Y are connected H -spaces of finite type and if there exists $T \in W_1$ such that $\Sigma X + T = \Sigma Y + T$, then there exists $A \in W$ such that $X \times A = Y \times A$.*

PROOF. Use (β) to get $\Omega \Sigma X \times \Omega T \times \Omega \Sigma(\Omega \Sigma X \wedge \Omega T) = \Omega \Sigma Y \times \Omega T \times \Omega \Sigma(\Omega \Sigma Y \wedge \Omega T)$.

By a theorem of Sugawara [6, p. 208], $\Omega \Sigma X = X \times \Omega \Sigma(X \wedge X)$ for any H -space X . The theorem now follows from Theorem 2.3 if we take $P = X, Q = Y, B = \Omega T$, and $\varphi = \Omega \Sigma(P_1 \wedge P_1) \times \Omega \Sigma(P_2 \wedge \Omega \Sigma P_1)$ and $\psi = P_1$.

We will also prove a partial converse to Theorem 2.4. We need two preparatory lemmas.

LEMMA 2.8. *Let $X, Y \in W_1, T \in W$. If $\Omega X \times T = \Omega Y \times T$ then for every $M \in W_1$ there exists $M' \in W$ such that $\Omega(X + M) \times M' = \Omega(Y + M) \times M'$.*

PROOF. Using (β) it is enough to prove that there exists M'' such that $\Omega \Sigma(\Omega X \wedge \Omega M) \times M'' = \Omega \Sigma(\Omega Y \wedge \Omega M) \times M''$. But that will follow from Theorem 2.3 by setting $P = \Omega X, Q = \Omega Y, B = T, \varphi = *, \psi = \Omega \Sigma(P_1 \wedge P_3)$ and $C = \Omega M$.

The following result is known to Ganea. We give a proof in the spirit of the arguments of this paper.

LEMMA 2.9. *If X is a simply connected H' -space of finite type, then there exists a suspension A of finite type such that $X + A$ is a suspension.*

PROOF. It is shown in [6, p. 209] that if X is an H' -space, then the standard map $r: \Sigma \Omega X \rightarrow X$ has a cross section $i: X \rightarrow \Sigma \Omega X$ and then $X + X_1 = \Sigma \Omega X$ where $X_1 = \Sigma \Omega X / X$. Then X_1 is an H' -space of finite type, being a retract of $\Sigma \Omega X$. Moreover, $\text{conn } X_1 > \text{conn } X$, since the map $r: \Sigma \Omega X \rightarrow X$ induces isomorphism of the first nonvanishing homotopy groups. Iterating the process we construct a sequence of H' -spaces X_i of finite type such that $X_0 = X, X_i + X_{i+1} = \Sigma \Omega X_i$, and $\text{conn } X_{i+1} > \text{conn } X_i$. If we set $A = \bigoplus_{i=1}^{\infty} X_i$, it is easy to see by arguments already used in the proof of Theorems 2.1 and 2.3 that both A and $X + A$ are suspensions, and that A is of finite type.

THEOREM 2.10. *If X, Y are 1-connected H' -spaces and $\Omega X \times T = \Omega Y \times T$ for some $T \in W$ then there exists $A \in W_1$ which is a suspension such that $X + A = Y + A$.*

PROOF. First assume X, Y are suspensions: $X = \Sigma X', Y = \Sigma Y'$. Apply (α) to get $\Sigma\Omega\Sigma X' + (\Sigma\Omega\Sigma X') \wedge T + \Sigma T = \Sigma\Omega\Sigma Y' + (\Sigma\Omega\Sigma Y') \wedge T + \Sigma T$. Using a theorem of James [9, p. 134] which asserts that $\Sigma\Omega\Sigma X' = \bigoplus_{i=1}^{\infty} \Sigma(X')^i$, and applying the same methods of Theorem 2.1 we get the theorem.

We now consider the general case. By Lemma 2.9, there exist X', Y' which are simply connected suspensions of finite type such that $X + X', Y + Y'$ are suspensions. Set $M = X' + Y'$. By using Lemma 2.8, we have that $\Omega(X + M) \times T' = \Omega(Y + M) \times T'$ for some $T' \in W$. As $X + M, Y + M$ are suspensions, we can apply what we have already proved to get $(X + M) + A' = (Y + M) + A'$ for some A' .

COROLLARY 2.11. *If $X + A = Y + A$ with X, Y, A simply connected of finite type then $\pi_*(X) \cong \pi_*(Y)$.*

PROOF. This is an easy corollary of Theorem 2.4, since there is cancellation for finitely generated abelian groups.

COROLLARY 2.12. *If h is a homology theory with $h_*(\text{point})$ of finite type, or $h_*(\text{point})$ a P.I.D. and if $X \times A = Y \times A$ with X, Y, A connected of finite type, then $h_*(X) \cong h_*(Y)$.*

PROOF. The assumption on h_* establishes that $h(X)$ is of finite type if X is of finite type, or that there is cancellation over $h_*(\text{point})$. The conclusion follows using Theorem 2.1.

Note that in the preceding theorem we did not use any Künneth theorem as was done in [4].

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