PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 2, July 1973

ELEMENTARY PROOFS OF TWO THEOREMS ON THE DISTRIBUTION OF NUMBERS $n\theta$ (mod 1)

ADOLF MIKLAVC

ABSTRACT. Simple and elementary proofs of denseness and uniform distribution of the numbers $\{n\theta\}$, θ irrational, on the unit interval.

1. **Introduction.** In this note we first show that any subinterval of the interval (0, 1) can be covered by a chain of equally separated points which belong to the set $S=\{P_n=\{n\theta\}\}$—θ being an irrational number ($0<\theta<1$). The separation ε between the points can be arbitrarily small. Using this property we can show in a simple way that the points $\{n\theta\}$ are uniformly distributed on the interval [0, 1] when $n\to\infty$.

2. **Proof.** Since no two points P_n coincide the sets must have at least one limit point in [0, 1]. For any arbitrarily small $\varepsilon>0$ we can therefore find two points $P_{n'}$ and $P_{n'+r}$ with $r>0$ which are separated from each other by less than ε. Let P_n be an arbitrary point in S. Then the points: $P_n, P_{n+r}, P_{n+2r}, \cdots, P_{n+ar}$, where a is an integer, form a chain of points separated by less than ε. From this we can see easily that there exist chains of equally separated points with the covering properties as stated in the Introduction.

We now use the conclusions we arrived at above to show that the points $\{n\theta\}$ are distributed uniformly. Consider two subintervals I_1 and I_2 of the interval (0, 1) which begin with the points $\{n_1\theta\}$ and $\{n_2\theta\}$ respectively and are of equal length. For definiteness we take $n_2-n_1=m>0$. Let $\eta_j(n_0)$ ($j=1, 2$) be the number of points $\{n\theta\}$ which are on the interval I_j if the sequence is terminated at $\{n_0\theta\}$. Choose an $n_0>n_1, n_2$ large enough so that $\eta_j(n_0)>m$. There are at least $[\eta_1(n_0)-m]$ points $\{n\theta\}\in I_1$ such that $n\leqq n_0-m$. For each of these points there is an "image" point $\{(n+m)\theta\}\in I_2$. Therefore

$$\eta_2(n_0) \geqq \eta_1(n_0) - m.$$

There are at least $[\eta_2(n_0)-m]$ points $\{n\theta\}\in I_2$ such that $\{(n-m)\theta\}\in I_1$.

Received by the editors July 3, 1972.

AMS (MOS) subject classifications (1970). Primary 10F40.

Therefore $\eta_1(n_0) \geqq \eta_2(n_0) - m$. Both inequalities give

$$m + \eta_1(n_0) \geqq \eta_2(n_0) \geqq \eta_1(n_0) - m.$$

In the limit $n_0 \to \infty$ we get $\eta_2(\infty)/\eta_1(\infty) = 1$.

Suppose that the intervals I_j do not begin with some point $\{n\theta\}$. We can cover them with the intervals $I_i(\varepsilon)$ which all begin with some $\{n_i\theta\}$ and are of equal length ε—ε being as small as we want. Let again $\eta_i(\varepsilon, n_0)$ be the number of points on the interval $I_i(\varepsilon)$ when the sequence is terminated at $\{n_0\theta\}$. Then we have for some integers N and b:

$$\sum_{i=1}^{N-1} \eta_i(\varepsilon, n_0) \leqq \eta_1(n_0) \leqq \sum_{i=1}^{N+1} \eta_i(\varepsilon, n_0)$$

and

$$\sum_{i=b+1}^{N+b-1} \eta_i(\varepsilon, n_0) \leqq \eta_2(n_0) \leqq \sum_{i=b+1}^{N+b+1} \eta_i(\varepsilon, n_0).$$

Since each of the intervals $I_i(\varepsilon)$ begins with some $\{n_i\theta\}$ we can find for any small $\delta > 0$ such n_0 that $|\eta_i(\varepsilon, n_0)/\eta_k(\varepsilon, n_0) - 1| \leqq \delta$ for any $\eta_i(\varepsilon, n_0)$ and $\eta_k(\varepsilon, n_0)$ in the sums. Therefore:

$$\eta_1(\varepsilon, n_0) \sum_{i=1}^{N-1} (1 - \delta) \leqq \eta_1(n_0) \leqq \eta_1(\varepsilon, n_0) \sum_{i=1}^{N+1} (1 + \delta).$$

The same is true for $\eta_2(n_0)$. For $n_0 \to \infty$, δ can be arbitrarily small and we have

$$\frac{N-1}{N+1} \leqq \frac{\eta_1(\infty)}{\eta_2(\infty)} \leqq \frac{N+1}{N-1}.$$

By choosing ε small enough we can make N as large as we please. Therefore $\eta_1(\infty)/\eta_2(\infty) = 1$. Since this holds for two arbitrary intervals of equal length it follows that the points $\{n\theta\}$ are uniformly distributed on the interval $[0, 1]$ when $n \to \infty$.

I am grateful to Professors J. R. Dorfman and E. M. Wright for their helpful comments.

BIBLIOGRAPHY

1. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Oxford Univ. Press, London, 1960.

2. I. Niven, *Diophantine approximations*, Interscience Tracts in Pure and Appl. Math., no. 14, Interscience, New York, 1963. MR **26** #6120.

DEPARTMENT OF PHYSICS AND ASTRONOMY, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742