ELEMEN{ARY PROOFS OF TWO THEOREMS ON THE DISTRIBUTION OF NUMBERS \( n\theta \) (mod 1)

ADOLF MIKLAVC

Abstract. Simple and elementary proofs of denseness and uniform distribution of the numbers \( \{n\theta\} \), \( \theta \) irrational, on the unit interval.

1. Introduction. In this note we first show that any subinterval of the interval \((0, 1)\) can be covered by a chain of equally separated points which belong to the set \( S = \{P_n = \{n\theta\}\} \) — \( \theta \) being an irrational number \((0 < \theta < 1)\). The separation \( \varepsilon \) between the points can be arbitrarily small. Using this property we can show in a simple way that the points \( \{n\theta\} \) are uniformly distributed on the interval [0, 1] when \( n \to \infty \).

2. Proof. Since no two points \( P_n \) coincide the sets must have at least one limit point in [0, 1]. For any arbitrarily small \( \varepsilon > 0 \) we can therefore find two points \( P_n \) and \( P_{n+r} \) with \( r > 0 \) which are separated from each other by less than \( \varepsilon \). Let \( P_n \) be an arbitrary point in \( S \). Then the points: \( P_n, P_{n+r}, P_{n+2r}, \ldots, P_{n+ar} \), where \( a \) is an integer, form a chain of points separated by less than \( \varepsilon \). From this we can see easily that there exist chains of equally separated points with the covering properties as stated in the Introduction.

We now use the conclusions we arrived at above to show that the points \( \{n\theta\} \) are distributed uniformly. Consider two subintervals \( I_1 \) and \( I_2 \) of the interval \((0, 1)\) which begin with the points \( \{n_1\theta\} \) and \( \{n_2\theta\} \) respectively and are of equal length. For definiteness we take \( n_2 - n_1 = m > 0 \). Let \( \eta_j(n_0) \) \((j=1, 2)\) be the number of points \( \{n\theta\} \) which are on the interval \( I_j \) if the sequence is terminated at \( \{n_0\theta\} \). Choose an \( n_0 > n_1, n_2 \) large enough so that \( \eta_j(n_0) > m \). There are at least \( [\eta_1(n_0) - m] \) points \( \{n\theta\} \in I_1 \) such that \( n \leq n_0 - m \). For each of these points there is an "image" point \( \{(n+m)\theta\} \in I_2 \). Therefore

\[
\eta_2(n_0) \geq \eta_1(n_0) - m.
\]

There are at least \( [\eta_2(n_0) - m] \) points \( \{n\theta\} \in I_2 \) such that \( \{(n-m)\theta\} \in I_1 \).

Received by the editors July 3, 1972.


© American Mathematical Society 1973

279
Therefore $\eta_1(n_0) \geq \eta_2(n_0) - m$. Both inequalities give

$$m + \eta_1(n_0) \geq \eta_2(n_0) \geq \eta_1(n_0) - m.$$ 

In the limit $n_0 \to \infty$ we get $\eta_2(\infty)/\eta_1(\infty) = 1$.

Suppose that the intervals $I_i$ do not begin with some point $\{n\theta\}$. We can cover them with the intervals $I_i(\varepsilon)$ which all begin with some $\{n\theta\}$ and are of equal length $\varepsilon - \varepsilon$ being as small as we want. Let again $\eta_i(e, n_0)$ be the number of points on the interval $I_i(e)$ when the sequence is terminated at $\{n_0\}$. Then we have for some integers $N$ and $b$:

$$\sum_{i=1}^{N-1} \eta_i(\varepsilon, n_0) \leq \eta_1(n_0) \leq \sum_{i=1}^{N+1} \eta_i(\varepsilon, n_0)$$

and

$$\sum_{i=b+1}^{N+b-1} \eta_i(\varepsilon, n_0) \leq \eta_2(n_0) \leq \sum_{i=b+1}^{N+b+1} \eta_i(\varepsilon, n_0).$$

Since each of the intervals $I_i(\varepsilon)$ begins with some $\{n\theta\}$ we can find for any small $\delta > 0$ such $n_0$ that $|\eta_i(e, n_0)/\eta_k(e, n_0) - 1| \leq \delta$ for any $\eta_i(e, n_0)$ and $\eta_k(e, n_0)$ in the sums. Therefore:

$$\eta_1(e, n_0) \sum_{i=1}^{N-1} (1 - \delta) \leq \eta_1(n_0) \leq \eta_1(e, n_0) \sum_{i=1}^{N+1} (1 + \delta).$$

The same is true for $\eta_2(n_0)$. For $n_0 \to \infty$, $\delta$ can be arbitrarily small and we have

$$\frac{N - 1}{N + 1} \leq \frac{\eta_1(\infty)}{\eta_2(\infty)} \leq \frac{l + 1}{N - 1}.$$ 

By choosing $\varepsilon$ small enough we can make $N$ as large as we please. Therefore $\eta_1(\infty)/\eta_2(\infty) = 1$. Since this holds for two arbitrary intervals of equal length it follows that the points $\{n\theta\}$ are uniformly distributed on the interval $[0, 1]$ when $n \to \infty$.

I am grateful to Professors J. R. Dorfman and E. M. Wright for their helpful comments.

**BIBLIOGRAPHY**


**Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742**