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THE RING OF POLYNOMIALS OVER A VON NEUMANN REGULAR RING

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ABSTRACT. It is shown that the ring of polynomials in one indeterminate over a commutative von Neumann regular ring with identity element is semihereditary.

Let R be a commutative von Neumann regular ring with identity element. Soublin stated without proof in **[4]**, and again in **[5]**, that the polynomial ring $R[X]$ is coherent. Recently, Sabbagh proved that a ring of polynomials in any number of indeterminates over R is coherent [**3**]. Carson obtained similar results for certain noncommutative rings [**1**]. In this note we prove the following result for the case of a single indeterminate.

THEOREM. *$R[X]$ is semihereditary.*

By a result of Vasconcelos [**6**, Theorem 4.2] the assertion follows if we show that w.gl.dim $R[X]\leqq 1$ and that the annihilator of each element of $R[X]$ is finitely generated.

Since w.gl.dim $R=0$, it is a consequence of a result of Jensen [**2**, Theorem 2] that w.gl.dim $R[X]=1$.

Let $f(X)=a_0+a_1X+\cdots+a_nX^n \in R[X]$ and let A be the annihilator of $f(X)$ in $R[X]$. For $i=0,\cdots,n$, let e_i be an idempotent element in R such that $Ra_i=Re_i$. Let $e=(1-e_0)\cdots(1-e_n)$. Clearly, $R[X]e\subseteq A$. Let $g(X)=b_0+b_1X+\cdots+b_kX^k \in A$: we show by double induction on k and n that $g(X)\in R[X]e$. Thus, $A=R[X]e$.

If $k=0$, then $f(X)b_0=0$ and so $e_ib_0=0$ for $i=0,\cdots,n$. Hence $b_0=b_0e\in R[X]e$

Suppose $k>0$ and that, for arbitrary n, if $h(X)\in A$ and deg $h(X)<k$, then $h(X)\in R[X]e$. If $n=0$, then $e_0g(X)=0$ and $g(X)=g(X)e\in R[X]e$. Suppose $n>0$ and that the inclusion we are asserting holds when $f(X)$ is replaced by a polynomial of degree less than n.

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Since $e_n b_k=0$ we have $\deg e_n g(X)<k$, and certainly $f(X)e_n g(X)=0$. Hence, by the induction assumption on k, there is a polynomial $m(X) \in R[X]$ such that $e_n g(X)=em(X)$. Then $e_n g(X)=e_n^2 g(X)=e_n em(X)=0$, and since a_n is a multiple of e_n it follows that $a_n g(X)=0$. Consequently,

$$(f(X) - a_n X^n)g(X) = 0,$$

and by the induction assumption on n,

$$g(X) \in R[X](1 - e_0) \cdots (1 - e_{n-1}).$$

Therefore, $g(X)=g(X)(1-e_n) \in R[X]e$.

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