

ON BOREL MEASURES AND BAIRE'S CLASS 3

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ABSTRACT. Let S be a complete and separable metric space and μ a σ -finite, complete Borel measure on S . Let Φ be the family of all real-valued functions, continuous μ -a.e. Let $B_\alpha(\Phi)$ be the functions of Baire's class α generated by Φ . It is shown that if μ is not a purely atomic measure whose set of atoms form a dispersed subset of S , then $B_2(\Phi) \neq B_{\omega_1}(\Phi)$, where ω_1 denotes the first uncountable ordinal.

If Φ is a family of real-valued functions defined on a set S , then, $B(\Phi)$, the Baire system of functions generated by Φ is the smallest subfamily of R^X which contains Φ and which is closed under the process of taking pointwise limits of sequences. The family $B(\Phi)$ can be generated from Φ as follows: Let $B_0(\Phi) = \Phi$ and for each ordinal α , let $B_\alpha(\Phi)$ be the family of all pointwise limits of sequences taken from $\bigcup_{\gamma < \alpha} B_\gamma(\Phi)$. Thus, $B_{\omega_1}(\Phi)$ is the Baire system of functions generated by Φ , where ω_1 is the first uncountable ordinal. The Baire order of a family Φ is the first ordinal α such that $B_\alpha(\Phi) = B_{\alpha+1}(\Phi)$.

Kuratowski has proved that if S is a metric space and Φ is the family of all real-valued functions on S which are continuous except for a first category set, then the order of Φ is 1 and $B_1(\Phi)$ is the family of all functions which have the Baire property in the wide sense [1, p. 323].

Let S be a complete separable metric space, let μ be a σ -finite, complete Borel measure on S and let Φ be the family of all real-valued functions on S , whose set of points of discontinuity is of μ -measure 0. In [3], it was shown that the order of Φ is 1 if and only if μ is purely atomic and the set of atoms of μ is a dispersed [7] (scattered) subset of S . Thus, as far as the Baire order problem is concerned the notion of first category and measure 0 cannot, in general, be interchanged.

The purpose of this paper is to prove that if μ is not a purely atomic measure whose atoms form a dispersed set, then the Baire order of Φ is at least 3. This will be accomplished by exhibiting a function in $B_3(C(S))$ which is not in $B_2(\Phi)$. Of course, $B_3(C(S))$ is a subfamily of $B_3(\Phi)$.

Received by the editors March 16, 1972 and, in revised form, October 18, 1972.
AMS (MOS) subject classifications (1970). Primary 28A05, 26A21; Secondary 02K30.
Key words and phrases. Borel measure, Baire function, Baire class α , dispersed set, ambiguous sets.

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In what follows suppose μ is not a purely atomic measure whose set of atoms form a dispersed set. In order to prove the main theorem the following technical lemma is employed.

LEMMA. *There is a perfect set M of finite measure such that if V is an open set intersecting M , then $\mu(M \cap V) > 0$ and there is a set of sequences, $\{T_{np}\}_{p=1}^{\infty}$, $n=1, 2, 3, \dots$, such that*

(1) $\{T_{1p}\}_{p=1}^{\infty}$ is a sequence of disjoint, perfect, nowhere dense subsets of M such that $T_1 = \bigcup_{p=1}^{\infty} T_{1p}$ is a dense subset of M and for each p , if V is an open set intersecting T_{1p} , then $\mu(T_{1p} \cap V) > 0$;

(2) for each n , $\{T_{n+1,p}\}_{p=1}^{\infty}$ is a sequence of disjoint perfect subsets of M such that for each k , $T_{n+1,k}$ is a subset of some term of $\{T_{np}\}_{p=1}^{\infty}$ and is nowhere dense with respect to that set, and if V is an open set intersecting $T_{n+1,k}$ then $\mu(T_{n+1,k} \cap V) > 0$; and

(3) for each n and p , the union of all the sets in $\{T_{n+1,k}\}_{k=1}^{\infty}$ which are subsets of T_{np} is dense in T_{np} .

PROOF. Since the support of μ is a closed set, it can be decomposed into a perfect part, P , and a countable scattered (dispersed) part D . From the assumptions made concerning μ , we have that the perfect part of this decomposition is nonempty.

It follows that there is a perfect subset M of P of finite measure such that if V is an open set which meets M , then $\mu(M \cap V) > 0$.

If H is a family of disjoint, perfect, nowhere dense subsets of M , each having positive μ -measure then the family H is countable. Let G be the collection of all such families H which have the additional property that if an open set V meets some member K of H , then $\mu(V \cap K) > 0$. Let the collection G be partially ordered by inclusion. Since every totally ordered subsystem of G has an upper bound, G has a maximal element: $\{T_{1p}\}_{p=1}^{\infty}$.

Now, suppose that $T_1 = \bigcup_{p=1}^{\infty} T_{1p}$ is not dense in M . Let U be an open set which meets M such that $\text{Cl } U$ does not intersect T_1 . The set $\text{Cl}(U \cap M)$ is a perfect subset of M and $\mu(\text{Cl}(U \cap M)) > 0$. It follows that there is a perfect nowhere subset K of M , lying in $\text{Cl}(U \cap M)$ such that if V is an open set which meets K , then $\mu(V \cap K) > 0$. This contradicts the maximality of the family $\{T_{1p}\}_{p=1}^{\infty}$. Thus, T_1 is dense in M .

Arguments, similar to the preceding one, can be given to complete the proof of the lemma.

THEOREM. *Let S be a complete separable metric space, let μ be a σ -finite, complete Borel measure on S and let Φ be the family of all real-valued functions on S which are continuous μ -a.e. If the Baire order of Φ is not 1, then the order of Φ is at least 3.*

PROOF. Let M be a perfect subset of S and $\{T_{np}\}_{p=1}^\infty, n=1, 2, 3, \dots$, a set of sequences of subsets of M satisfying the conditions of the conclusion of the lemma. Let $T = \bigcap_{n=1}^\infty T_n$. The set T is an $F_{\sigma\delta}$ set and \mathcal{E}_T , the characteristic function of T is in Baire's class 3 ([8], [9]).

Suppose \mathcal{E}_T is in $B_2(\Phi)$. By Theorem 3 of [4], there is an F_σ set K such that $\mu(K)=0$ and a function g in Baire's class 2 such that if x is in $S-K$, $g(x)=\mathcal{E}_T(x)$. Hence, $\mathcal{E}_T|K'$ is in Baire's class 2 with respect to K' , the complement of K . So, $T \cap K'$ is a $G_{\delta\sigma}$ set with respect to K' . There is a $G_{\delta\sigma}$ subset A of S such that $A \cap K' = T \cap K'$. Since K' is a G_δ set in S , $T \cap K'$ is a $G_{\delta\sigma}$ set in S .

We have $T' = T'_1 \cup (T_1 - T_2) \cup (T_2 - T_3) \cup \dots$. Since $T_n - T_{n+1}$ is a $G_{\delta\sigma}$ set, for each n and T'_1 and K are $G_{\delta\sigma}$ sets, it follows that $T' \cup K$, the complement of $T \cap K'$, is a $G_{\delta\sigma}$ set. Thus, $T' \cup K$ is an ambiguous set of class 2.

It follows that there is a sequence $\{A_n\}_{n=1}^\infty$ of ambiguous sets of class 1 such that

$$T' \cup K = \bigcup_{n=1}^\infty (A_n \cap A_{n+1} \cap \dots) = \bigcap_{n=1}^\infty (A_n \cup A_{n+1} \cup \dots) \quad [6, \text{p. 355}].$$

Let $\{A_{n_i}\}_{i=1}^\infty$ be the subsequence of $\{A_n\}_{n=1}^\infty$ consisting of all terms which intersect T' and for each $k > 1$, let $\{A_{n_k i}\}_{i=1}^\infty$ be the subsequence of $\{A_n\}_{n=1}^\infty$ consisting of all terms which intersect $T_{k-1} - T_k$ and having subscripts $\geq k$.

For each k , let $B_k = \bigcup_{i=1}^\infty A_{n_k i}$. It follows that for each k , B_k is an F_σ set, B_1 contains T'_1 and if $k > 1$, B_k contains $T_{k-1} - T_k$. Also, it follows that $\limsup_{k \rightarrow \infty} B_k$ is a subset of $\limsup_{n \rightarrow \infty} A_n$, which is $T' \cup K$.

Let $K = \bigcup_{n=1}^\infty F_n$, where for each n , F_n is a closed set of measure 0 and F_{n+1} contains F_n .

Since B_1 is an F_σ set containing T'_1 , and T_1 is of the first category with respect to P , there is an open set C_1 intersecting P such that $\text{Cl}(C_1 \cap P)$ is a subset of B_1 . Since F_1 is closed and $\mu(F_1)=0$, $F_1 \cap P$ is a closed, nowhere dense subset of P . Let S_1 be a spherical ball of radius less than 1, intersecting P such that $\text{Cl}(S_1 \cap P)$ is a subset of $C_1 \cap P$ and $\text{Cl}(S_1 \cap P)$ does not intersect F_1 .

Since T_1 is a dense subset of P , there is a positive integer n_1 such that T_{1n_1} , intersects S_1 . Thus, $S_1 \cap T_{1n_1}$, is a dense in itself subset of T_{1n_1} , and $H_1 = \text{Cl}(S_1 \cap T_{1n_1})$, is a perfect subset of T_{1n_1} , such that if \mathcal{O} is an open set intersecting H_1 , then $\mu(\mathcal{O} \cap H_1) > 0$.

As B_2 is an F_σ set containing $T_1 - T_2$ and T_2 is of the first category with respect to H_1 , B_2 is not of the first category with respect to H_1 . There is an open set C_2 lying in S_1 and intersecting H_1 such that $\text{Cl}(H_1 \cap C_2)$ is a subset of B_2 . Since F_2 is closed and $\mu(F_2)=0$, $F_2 \cap H_1$ is a closed, nowhere dense subset of H_1 . Let S_2 be a spherical ball of radius less than $\frac{1}{2}$

intersecting H_1 such that $\text{Cl}(S_2)$ subset of C_2 and $\text{Cl}(S_2 \cap H_1)$ does not intersect F_2 .

As $T_2 \cap T_{1n_1}$ is a dense subset of T_{1n_1} , there is a positive integer n_2 such that T_{2n_2} intersects $S_2 \cap H_1$. Then T_{2n_2} is a subset of T_{1n_1} , $S_2 \cap T_{2n_2}$ is a dense in itself subset of $S_2 \cap H_1$ and $H_2 = \text{Cl}(S_2 \cap T_{2n_2})$ is a perfect subset of T_{2n_2} such that if O is an open set intersecting it, then $\mu(O \cap H_2) > 0$.

Suppose $k > 1$ and sets C_i , S_i , and T_{in_i} , $1 \leq i \leq k$, have been defined having the following properties:

- (1) C_k is an open set lying in S_{k-1} such that $\text{Cl}(H_{k-1} \cap C_k)$, where $H_{k-1} = \text{Cl}(S_{k-1} \cap T_{k-1n_{k-1}})$, is a subset of B_k ;
- (2) S_k is a spherical ball of radius less than $1/k$ intersecting H_{k-1} such that $\text{Cl}(S_k)$ is a subset of C_k and $\text{Cl}(S_k \cap H_{k-1})$ does not intersect F_k ;
- (3) n_k is a positive integer such that T_{kn_k} intersects $S_k \cap H_{k-1}$.

Now, an argument analogous to the one given above to obtain the sets C_2 , S_2 , and T_{2n_2} may be used to obtain sets C_{k+1} , S_{k+1} , and $T_{k+1n_{k+1}}$ having properties (1), (2), and (3) listed above where $k+1$ is substituted for k .

The sequence $\{H_p\}_{p=1}^\infty$ is a monotonically decreasing sequence of closed point sets in the complete and separable space S and the diameter of H_{p+1} is less than $2/(p+1)$. There is a point w common to all the terms of the sequence $\{H_p\}_{p=1}^\infty$. As for each p , H_p is a subset of B_p and of T_p and H_p does not intersect F_p , w is in $\limsup B_n$ and w is in $T \cap K'$. But, $\limsup B_n$ is a subset of $T' \cup K$. This is a contradiction. This completes the argument for the theorem.

L. Kantorovitch has shown that in the special case $S = [0, 1]$ and μ is Lebesgue measure, there is a function in Baire's class 2, not in $B_1(\mathcal{F})$ [10]. The theorem proved in this paper shows that there is a function in Baire's class 3, not in $B_2(\mathcal{F})$, if μ is not a purely atomic measure having a dispersed set of atoms. It is not difficult to show from results in [4] that $B_{\alpha+1}(\mathcal{F}) \neq B_\alpha(\mathcal{F})$ if and only if there is a function f in Baire's class $\alpha+1$ such that if g is in Baire's class α , then the set $(f \neq g)$ is not a subset of an F_σ set of measure 0.

CONJECTURE. If the Baire order of Φ , the family of all real-valued functions continuous a.e. is not 0 or 1, then it is ω_1 .

REMARK. To settle this question, calls for some delicate analysis as it is well known that every measurable function f agrees with a function g in Baire's class 2 almost everywhere; however, as has been shown here the topological nature of the set $(f \neq g)$ is very important in this process.

QUESTION. Is there any family Φ whose Baire order is not 0, 1, 2 or ω_1 ?

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