ON BOREL MEASURES AND BAIRE'S CLASS 3

R. DANIEL MAULDIN

Abstract. Let $S$ be a complete and separable metric space and $\mu$ a $\sigma$-finite, complete Borel measure on $S$. Let $\Phi$ be the family of all real-valued functions, continuous $\mu$-a.e. Let $B_\alpha(\Phi)$ be the functions of Baire's class $\alpha$ generated by $\Phi$. It is shown that if $\mu$ is not a purely atomic measure whose set of atoms form a dispersed subset of $S$, then $B_\alpha(\Phi) \neq B_{\omega_1}(\Phi)$, where $\omega_1$ denotes the first uncountable ordinal.

If $\Phi$ is a family of real-valued functions defined on a set $S$, then, $B(\Phi)$, the Baire system of functions generated by $\Phi$ is the smallest subfamily of $\mathbb{R}^X$ which contains $\Phi$ and which is closed under the process of taking pointwise limits of sequences. The family $B(\Phi)$ can be generated from $\Phi$ as follows: Let $B_0(\Phi) = \Phi$ and for each ordinal $\alpha$, let $B_\alpha(\Phi)$ be the family of all pointwise limits of sequences taken from $\bigcup_{\gamma < \alpha} B_\gamma(\Phi)$. Thus, $B_{\omega_1}(\Phi)$ is the Baire system of functions generated by $\Phi$, where $\omega_1$ is the first uncountable ordinal. The Baire order of a family $\Phi$ is the first ordinal $\alpha$ such that $B_\alpha(\Phi) = B_{\omega_1}(\Phi)$.

Kuratowski has proved that if $S$ is a metric space and $\Phi$ is the family of all real-valued functions on $S$ which are continuous except for a first category set, then the order of $\Phi$ is 1 and $B_1(\Phi)$ is the family of all functions which have the Baire property in the wide sense [1, p. 323].

Let $S$ be a complete separable metric space, let $\mu$ be a $\sigma$-finite, complete Borel measure on $S$ and let $\Phi$ be the family of all real-valued functions on $S$, whose set of points of discontinuity is of $\mu$-measure 0. In [3], it was shown that the order of $\Phi$ is 1 if and only if $\mu$ is purely atomic and the set of atoms of $\mu$ is a dispersed [7] (scattered) subset of $S$. Thus, as far as the Baire order problem is concerned the notion of first category and measure 0 cannot, in general, be interchanged.

The purpose of this paper is to prove that if $\mu$ is not a purely atomic measure whose atoms form a dispersed set, then the Baire order of $\Phi$ is at least 3. This will be accomplished by exhibiting a function in $B_3(C(S))$ which is not in $B_2(\Phi)$. Of course, $B_3(C(S))$ is a subfamily of $B_3(\Phi)$.

Received by the editors March 16, 1972 and, in revised form, October 18, 1972. 
Key words and phrases. Borel measure, Baire function, Baire class $\alpha$, dispersed set, ambiguous sets.
In what follows suppose $\mu$ is not a purely atomic measure whose set of atoms form a dispersed set. In order to prove the main theorem the following technical lemma is employed.

**Lemma.** There is a perfect set $M$ of finite measure such that if $V$ is an open set intersecting $M$, then $\mu(M \cap V) > 0$ and there is a set of sequences, $\{T_{np}\}_{n=1}^{\infty}, n=1, 2, 3, \ldots$, such that

1. $\{T_{1p}\}_{p=1}^{\infty}$ is a sequence of disjoint, perfect, nowhere dense subsets of $M$ such that $T_1 = \bigcup_{p=1}^{\infty} T_{1p}$ is a dense subset of $M$ and for each $p$, if $V$ is an open set intersecting $T_{1p}$, then $\mu(T_{1p} \cap V) > 0$;

2. for each $n$, $\{T_{n+1,p}\}_{p=1}^{\infty}$ is a sequence of disjoint perfect subsets of $M$ such that for each $k$, $T_{n+1,k}$ is a subset of some term of $\{T_{np}\}_{p=1}^{\infty}$ and is nowhere dense with respect to that set, and if $V$ is an open set intersecting $T_{n+1,k}$ then $\mu(T_{n+1,k} \cap V) > 0$; and

3. for each $n$ and $p$, the union of all the sets in $\{T_{n+1,k}\}_{k=1}^{\infty}$ which are subsets of $T_{np}$ is dense in $T_{np}$.

**Proof.** Since the support of $\mu$ is a closed set, it can be decomposed into a perfect part, $P$, and a countable scattered (dispersed) part $D$. From the assumptions made concerning $\mu$, we have that the perfect part of this decomposition is nonempty.

It follows that there is a perfect subset $M$ of $P$ of finite measure such that $\mu(M \cap V) > 0$.

If $H$ is a family of disjoint, perfect, nowhere dense subsets of $M$, each having positive $\mu$-measure then the family $H$ is countable. Let $G$ be the collection of all such families $H$ which have the additional property that if an open set $V$ meets some member $K$ of $H$, then $\mu(V \cap K) > 0$. Let the collection $G$ be partially ordered by inclusion. Since every totally ordered subsystem of $G$ has an upper bound, $G$ has a maximal element: $\{T_{1p}\}_{p=1}^{\infty}$.

Now, suppose that $T_1 = \bigcup_{p=1}^{\infty} T_{1p}$ is not dense in $M$. Let $U$ be an open set which meets $M$ such that $\text{Cl } U$ does not intersect $T_1$. The set $\text{Cl}(U \cap M)$ is a perfect subset of $M$ and $\mu(\text{Cl}(U \cap M)) > 0$. It follows that there is a perfect nowhere subset $K$ of $M$, lying in $\text{Cl}(U \cap M)$ such that if $V$ is an open set which meets $K$, then $\mu(V \cap K) > 0$. This contradicts the maximality of the family $\{T_{1p}\}_{p=1}^{\infty}$. Thus, $T_1$ is dense in $M$.

Arguments, similar to the preceding one, can be given to complete the proof of the lemma.

**Theorem.** Let $S$ be a complete separable metric space, let $\mu$ be a $\sigma$-finite, complete Borel measure on $S$ and let $\Phi$ be the family of all real-valued functions on $S$ which are continuous $\mu$-a.e. If the Baire order of $\Phi$ is not 1, then the order of $\Phi$ is at least 3.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Proof. Let $M$ be a perfect subset of $S$ and $\{T_{n}\}_{n=1}^{\infty}$, $n=1, 2, 3, \ldots$, a set of sequences of subsets of $M$ satisfying the conditions of the conclusion of the lemma. Let $T=\bigcap_{n=1}^{\infty} T_{n}$. The set $T$ is an $F_{\sigma}$ set and $\mathcal{C}_{T}$, the characteristic function of $T$ is in Baire's class 3 ([8], [9]).

Suppose $\mathcal{C}_{T}$ is in $B_{2}(\Phi)$. By Theorem 3 of [4], there is an $F_{\sigma}$ set $K$ such that $\mu(K)=0$ and a function $g$ in Baire's class 2 such that if $x$ is in $S-K$, $g(x)=\mathcal{C}_{T}(x)$. Hence, $\mathcal{C}_{T}|K'$ is in Baire's class 2 with respect to $K'$, the complement of $K$. So, $T\cap K'$ is a $G_{\delta\sigma}$ set with respect to $K'$. There is a $G_{\delta\sigma}$ subset $A$ of $S$ such that $A\cap K'=T\cap K'$. Since $K'$ is a $G_{\delta}$ set in $S$, $T\cap K'$ is a $G_{\sigma}$ set in $S$.

We have $T'=T_{1}'\cup(T_{2}'-T_{3}')\cup(T_{2}'-T_{3}')\cup\cdots$. Since $T_{n}-T_{n+1}$ is a $G_{\delta\sigma}$ set, for each $n$ and $T_{1}'$ and $K$ are $G_{\delta\sigma}$ sets, it follows that $T'\cap K$, the complement of $T\cap K'$, is a $G_{\delta\sigma}$ set. Thus, $T'\cap K$ is an ambiguous set of class 2.

It follows that there is a sequence $\{A_{n}\}_{n=1}^{\infty}$ of ambiguous sets of class 1 such that

$$T'\cup K = \bigcup_{n=1}^{\infty} (A_{n} \cap A_{n+1} \cap \cdots) = \bigcap_{n=1}^{\infty} (A_{n} \cup A_{n+1} \cup \cdots)$$

[6, p. 355].

Let $\{A_{n1}\}_{i=1}^{\infty}$ be the subsequence of $\{A_{n}\}_{n=1}^{\infty}$ consisting of all terms which intersect $T'$ and for each $k>1$, let $\{A_{nk}\}_{i=1}^{\infty}$ be the subsequence of $\{A_{n}\}_{n=1}^{\infty}$ consisting of all terms which intersect $T_{k-1}-T_{k}$ and having subscript $\geq k$.

For each $k$, let $B_{k}=\bigcup_{i=1}^{\infty} A_{nk}$. It follows that for each $k$, $B_{k}$ is an $F_{\sigma}$ set, $B_{k}$ contains $T_{1}'$ and if $k>1$, $B_{k}$ contains $T_{k-1}-T_{k}$. Also, it follows that $\limsup_{k\rightarrow\infty} B_{k}$ is a subset of $\limsup_{n\rightarrow\infty} A_{n}$, which is $T'\cap K$.

Let $K=\bigcup_{n=1}^{\infty} F_{n}$, where for each $n$, $F_{n}$ is a closed set of measure 0 and $F_{n+1}$ contains $F_{n}$.

Since $B_{1}$ is an $F_{\sigma}$ set containing $T_{1}'$, and $T_{1}'$ is of the first category with respect to $P$, there is an open set $C_{1}$ intersecting $P$ such that $C_{1}(C_{1}\cap P)$ is a subset of $B_{1}$. Since $F_{1}$ is closed and $\mu(F_{1})=0$, $F_{1}\cap P$ is a closed, nowhere dense subset of $P$. Let $S_{1}$ be a spherical ball of radius less than 1, intersecting $P$ such that $C_{1}(S_{1}\cap P)$ is a subset of $C_{1}\cap P$ and $C_{1}(S_{1}\cap P)$ does not intersect $F_{1}$.

Since $T_{1}$ is a dense subset of $P$, there is a positive integer $n_{1}$ such that $T_{1n_{1}} \cap S_{1} \subset T_{1}$ intersects $S_{1}$. Thus, $S_{1}\cap T_{1n_{1}}$ is a dense in itself subset of $T_{1n_{1}}$, and $H_{1}=C_{1}(S_{1}\cap T_{1n_{1}})$, is a perfect subset of $T_{1n_{1}}$ such that if $\emptyset$ is an open set intersecting $H_{1}$, then $\mu(\emptyset\cap H_{1})>0$.

As $B_{2}$ is an $F_{\sigma}$ set containing $T_{1}-T_{2}$ and $T_{2}$ is of the first category with respect to $H_{1}$, $B_{2}$ is not of the first category with respect to $H_{1}$. There is an open set $C_{2}$ lying in $S_{1}$ and intersecting $H_{1}$ such that $C_{1}(H_{1}\cap C_{2})$ is a subset of $B_{2}$. Since $F_{2}$ is closed and $\mu(F_{2})=0$, $F_{2}\cap H_{1}$ is a closed, nowhere dense subset of $H_{1}$. Let $S_{2}$ be a spherical ball of radius less than $\frac{1}{2}$.
intersecting \( H_1 \) such that \( \text{Cl}(S_2) \) subset of \( C_2 \) and \( \text{Cl}(S_2 \cap H_1) \) does not intersect \( F_2 \).

As \( T_2 \cap T_{1n^1} \) is a dense subset of \( T_{1n^1} \), there is a positive integer \( n_2 \) such that \( T_{2n^2} \) intersects \( S_2 \cap H_1 \). Then \( T_{2n^2} \) is a subset of \( T_{1n^1} \), \( S_2 \cap T_{2n^2} \) is a dense in itself subset of \( S_2 \cap H_1 \) and \( H_k = \text{Cl}(S_2 \cap T_{2n^2}) \) is a perfect subset of \( T_{2n^2} \) such that if \( O \) is an open set intersecting it, then \( \mu(O \cap H_2) > 0 \).

Suppose \( k > 1 \) and sets \( C_i, S_i, \) and \( T_{in^1} \), \( 1 \leq i \leq k \), have been defined having the following properties:

1. \( C_k \) is an open set lying in \( S_{k-1} \) such that \( \text{Cl}(H_{k-1} \cap C_k) \), where \( H_{k-1} = \text{Cl}(S_{k-1} \cap T_{k-1n^1}) \), is a subset of \( B_k \);
2. \( S_k \) is a spherical ball of radius less than \( 1/k \) intersecting \( H_{k-1} \) such that \( \text{Cl}(S_k) \) is a subset of \( C_k \) and \( \text{Cl}(S_k \cap H_{k-1}) \) does not intersect \( F_k \);
3. \( n_k \) is a positive integer such that \( T_{kn^k} \) intersects \( S_k \cap H_{k-1} \).

Now, an argument analogous to the one given above to obtain the sets \( C_2, S_2, \) and \( T_{2n^2} \) may be used to obtain sets \( C_{k+1}, S_{k+1}, \) and \( T_{k+1n^1} \) having properties (1), (2), and (3) listed above where \( k+1 \) is substituted for \( k \).

The sequence \( \{H_p\}_{p=1}^{\infty} \) is a monotonically decreasing sequence of closed point sets in the complete and separable space \( S \) and the diameter of \( H_{p+1} \) is less than \( 2/(p+1) \). There is a point \( w \) common to all the terms of the sequence \( \{H_p\}_{p=1}^{\infty} \). As for each \( p \), \( H_p \) is a subset of \( B_p \) and \( T_p \) and \( H_p \) does not intersect \( F_p \), \( w \) is in \( \lim \sup B_n \) and \( w \) is in \( T \cap K' \). But, \( \lim \sup B_n \) is a subset of \( T' \cup K \). This is a contradiction. This completes the argument for the theorem.

L. Kantorovitch has shown that in the special case \( S=[0,1] \) and \( \mu \) is Lebesgue measure, there is a function in Baire’s class 2, not in \( B_1(F) \) [10]. The theorem proved in this paper shows that there is a function in Baire’s class 3, not in \( B_2(F) \), if \( \mu \) is not a purely atomic measure having a dispersed set of atoms. It is not difficult to show from results in [4] that \( B_{x+1}(F) \neq B_x(F) \) if and only if there is a function \( f \) in Baire’s class \( x+1 \) such that if \( g \) is in Baire’s class \( x \), then the set \( (f \neq g) \) is not a subset of an \( F_x \) set of measure 0.

**Conjecture.** If the Baire order of \( \Phi \), the family of all real-valued functions continuous a.e. is not 0 or 1, then it is \( \omega_1 \).

**Remark.** To settle this question, calls for some delicate analysis as it is well known that every measurable function \( f \) agrees with a function \( g \) in Baire’s class 2 almost everywhere; however, as has been shown here the topological nature of the set \( (f \neq g) \) is very important in this process.

**Question.** Is there any family \( \Phi \) whose Baire order is not 0, 1, 2 or \( \omega_1 \)?

**References**


Department of Mathematics, University of Florida, Gainesville, Florida 32601