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AN EXTREMAL PROBLEM FOR THE GEOMETRIC MEAN OF POLYNOMIALS

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ABSTRACT. Let $M_{0,n}$ be the maximum of the geometric mean of all nth degree polynomials $\sum^n a_k e^{ikt}$ which satisfy $|a_k|=1$, $k=0, 1, \cdots, n$. We show the existence of certain polynomials R_n whose geometric mean is asymptotic to $\sqrt{n}$, thus proving that $M_{0,n}$ is itself asymptotic to $\sqrt{n}$.

Consider the class $\mathscr{P}_n$ of all nth degree polynomials $\sum_{k=0}^n a_k z^k$ for which $|a_k|=1$, $k=0, 1, \cdots, n$. Following the usual notation, let

$$M_r(f) = \left((2\pi)^{-1}\int_{-\pi}^{\pi} |f(e^{i\theta})|^r \, d\theta\right)^{1/r}$$

for $r>0$, and let

$$M_0(f) = G(f) = \exp\left((2\pi)^{-1}\int_{-\pi}^{\pi} \log |f(e^{i\theta})| \, d\theta\right),$$

so that, in particular, for $r=0$, 1, and 2, $M_r(f)$ is the geometric mean, arithmetic mean, and mean square of f, respectively.

Now, for $f \in \mathscr{P}_n$, and $0 \leqq r < 2$, we have $M_r(f) \leqq M_2(f) = (n+1)^{1/2}$. The question arises, how close can $M_r(f)$ be to $(n+1)^{1/2}$, the mean square? To formulate our question in more precise terms, let

$$M_{r,n} = \max_{\{f\}} M_r(f) \qquad (f \in \mathscr{P}_n)$$

for $r<2$, so that $M_{r,n} \leqq (n+1)^{1/2}$. For each $r<2$, we now ask: Is $M_{r,n}$ asymptotic to $\sqrt{n}$ as $n \to \infty$?

Newman [3] constructed $P_n \in \mathscr{P}_n$ for which[2] $M_4^4(P_n) = n^2 + O(n^{3/2})$, and thereby proved that $M_1(P_n) > \sqrt{n} - c$. Beller [1] noted that this

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[2] J. E. Littlewood [2], working with similar polynomials F_n, obtained a sharper estimate for $M_4^4(F_n)$, namely, $M_4^4(F_n) = n^2 + kn^{3/2} + O(n)$, where $k = \sqrt{2} - (2/\pi)(\sqrt{2} - 1)$.

implies the following, more complete result: $\boldsymbol{M}_{r,n} > \sqrt{n} - c/r$, for $0 < r < 2$ (c an absolute constant), thus answering our question affirmatively for all $r < 2$ except $r = 0$.

It is the purpose of this paper to show that $\boldsymbol{M}_{0,n} \sim \sqrt{n}$. It may be noted that this result yields an interesting corollary concerning the zeros of polynomials. Since, for all $f \in \mathscr{P}_n$, the product of *all* the zeros of $f(z)$ is numerically equal to 1, it follows from Jensen's theorem that $G(f) = \prod_i |r_i|$, where the r_i are the zeros of $f(z)$ for which $|r_i| > 1$. Thus, our result can be rephrased as follows: The maximum—taken over all $f \in \mathscr{P}_n$—of the product of the moduli of the zeros of $f(z)$ outside the unit circle, is asymptotic to $\sqrt{n}$.

Before proceeding with our theorem, let us review the situation for $r > 2$. Since, for such r, $M_r(f) \geqq (n+1)^{1/2}$, we consider

$$\boldsymbol{m}_{r,n} = \min_{\{f\}} M_r(f) \qquad (f \in \mathscr{P}_n).$$

We also define $M_\infty(f) = \sup_{\{\theta\}} |f(e^{i\theta})|$. Beller [1] has shown that $\boldsymbol{m}_{r,n}$ is asymptotic to $\sqrt{n}$, for $2 < r < \infty$. This, taken together with our present result, gives us an affirmative answer to our question for all $r \geqq 0$, except $r = \infty$, which still remains unsettled.

We now state our

THEOREM. *$\boldsymbol{M}_{0,n} \sim \sqrt{n}$. In fact, $\boldsymbol{M}_{0,n} > \sqrt{n} - c \log n$, where c is an absolute constant.*

PROOF. Choose $P_n \in \mathscr{P}_n$ which satisfy

$$M_4(P_n) = \sqrt{n} + O(1), \tag{1}$$

e.g., the P_n which Newman [3] constructed. Now, in estimating $G(P_n)$, one encounters difficulty where $|P_n|$ is close to zero. We will show that, by appropriately changing the constant term of P_n, this difficulty is eliminated.

Let $f^+(x) = f(x)$ $(f(x) \geqq 0)$; $f^+(x) = 0$ $(f(x) < 0)$. We will need the following well-known inequality:

$$(2\pi)^{-1} \int_{-\pi}^{\pi} \log |e^{it} + \alpha| \, dt = \log^+ |\alpha|, \tag{2}$$

where α is any complex number.

Now, let $Q_n(z) = P_n(z) - a_0$, where a_0 is the constant term of $P_n(z)$. By reversing the order of integration and applying (2), we find that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |e^{it} + Q_n(e^{i\theta})| \, d\theta \right) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |Q_n(e^{i\theta})| \, d\theta.$$

Thus, by the First Mean Value Theorem, there exists a real δ such that

$$(2\pi)^{-1}\int_{-\pi}^{\pi}\log|e^{i\delta}+Q_n(e^{i\theta})|\,d\theta=(2\pi)^{-1}\int_{-\pi}^{\pi}\log^+|Q_n(e^{i\theta})|\,d\theta. \tag{3}$$

Let $R_n(z)=e^{i\delta}+Q_n(z)$, so that $R_n\in\mathscr{P}_n$. Our Theorem now follows directly from the following

PROPOSITION. *$G(R_n)>\sqrt{n}-c\log n$, where c is an absolute constant.*

PROOF OF PROPOSITION. By Minkowski's inequality we have $M_4(P_n)-1\leqq M_4(Q_n)\leqq M_4(P_n)+1$, and this, combined with (1), yields

$$(2\pi)^{-1}\int_{-\pi}^{\pi}|Q_n|^4\,d\theta=n^2+O(n^{3/2}). \tag{4}$$

We also have

$$(2\pi)^{-1}\int_{-\pi}^{\pi}|Q_n|^2\,d\theta=n. \tag{5}$$

We now need the following lemma, which is actually a reverse arithmetic-geometric inequality:

LEMMA 1. *If $f(x)\geqq 0$, $\int_a^b f(x)\,dx=\lambda>0$, and $\int_a^b f^2(x)\,dx=\mu$, then*

$$\int_a^b\log(1+f(x))\,dx\geqq(\lambda^2/\mu)\log(1+\mu/\lambda).$$

PROOF. Set $p(x)=(1/\lambda)f(x)$, so that $p(x)$ is a nonnegative weight function. Let $\mu/\lambda=m$, so that $\int_a^b f(x)p(x)\,dx=m$. In these terms, we want to show that

$$\int_a^b\frac{\log(1+f(x))}{f(x)}p(x)\,dx\geqq(1/m)\log(1+m).$$

But direct differentiation shows that $\phi(t)=(1/t)\log(1+t)$ is convex on $[0,\infty)$, so that we may apply Jensen's inequality (see Zygmund [**4**, Vol. I, p. 24]), namely,

$$\int_a^b\phi(f(x))p(x)\,dx\geqq\phi\left(\int_a^b f(x)p(x)\,dx\right). \qquad \text{Q.E.D.}$$

REMARK. Since the function $(\lambda^2/\mu)\log(1+\mu/\lambda)$ is increasing in λ, for $\lambda>0$, and decreasing in μ, the conclusion of Lemma 1 still holds under the weaker hypothesis $f(x)\geqq 0$, $\int f(x)\,dx\geqq\lambda>0$, $\int f^2(x)\,dx\leqq\mu$.

LEMMA 2. *If* $\int_0^1 F(x)\,dx \geqq A > 1$ *and* $\int_0^1 F^2(x)\,dx \leqq B$, *then*[3]

$$\int_0^1 \log^+ F(x)\,dx \geqq \frac{(A-1)^2}{B-2A+1}\log\left(\frac{B-A}{A-1}\right).$$

PROOF. We apply our previous remark to $f(x)=(F(x)-1)^+$, so that

$$\int_0^1 f(x)\,dx \geqq \int_0^1 (F(x)-1)\,dx \geqq A-1>0$$

and

$$\int_0^1 f^2(x)\,dx \leqq \int_0^1 (F(x)-1)^2\,dx \leqq B-2A+1,$$

and we conclude that

$$\int_0^1 \log^+ F(x)\,dx = \int_0^1 \log(1+f(x))\,dx \geqq \frac{(A-1)^2}{B-2A+1}\log\left(\frac{B-A}{A-1}\right),$$

as required.

Now let $F(x)=|Q_n(e^{2\pi ix})|^2$. In terms of F, (4) and (5) become

$$\int_0^1 F(x)\,dx = n;\quad \int_0^1 F^2(x)\,dx \leqq n^2 + c_1 n^{3/2},$$

where c_1 is some absolute constant. Applying Lemma 2, we have

$$\text{(6)}\quad \begin{aligned}(2\pi)^{-1}\int_{-\pi}^{\pi} \log^+ |Q_n(e^{i\theta})|\,d\theta &= \frac{1}{2}\int_0^1 \log^+ F(x)\,dx\\ &> \frac{1}{2}\left(\frac{1-2/n}{1+c_1/\sqrt{n}}\right)\log n > \tfrac{1}{2}\log n - \left(\frac{c}{\sqrt{n}}\right)\log n,\end{aligned}$$

where c is another constant.

Combining (3) and (6), we conclude that

$$\begin{aligned}G(R_n) &= \exp\left((2\pi)^{-1}\int_{-\pi}^{\pi} \log^+ |Q_n(e^{i\theta})|\,d\theta\right)\\ &\geqq \sqrt{n}\exp(-(c/\sqrt{n})\log n) > \sqrt{n} - c\log n,\end{aligned}$$

thus proving the Proposition.

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