

DIRICHLET PROBLEMS FOR SINGULAR ELLIPTIC EQUATIONS

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ABSTRACT. Boundary value problems are formulated for the equation

$$(*) \quad L[u] = \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1} b_i \frac{\partial u}{\partial x_i} + \frac{h}{x_n} \frac{\partial u}{\partial x_n} + cu = f$$

in a bounded domain G in E_n with boundary $\partial G = S_1 \cup S_2$ where S_1 is in $x_n = 0$ and S_2 is in $x_n > 0$. A uniqueness theorem is established for (*) when boundary data is only given on S_2 for

$$h(x_1, \dots, x_{n-1}, 0) \geq 1;$$

whereas an existence and uniqueness theorem for the Dirichlet problem is proved for $h(x_1, x_2, \dots, x_{n-1}, 0) < 1$.

Let G be a bounded domain in a half space $x_n > 0$ with boundary $\partial G = S_1 \cup S_2$ where S_1 is contained in the hyperplane $x_n = 0$ and S_2 lies entirely in $x_n > 0$. We consider a class of second order linear partial differential equations

$$(1) \quad L[u] = \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1} b_i \frac{\partial u}{\partial x_i} + \frac{h}{x_n} \frac{\partial u}{\partial x_n} + cu = f$$

where the coefficients $a_{ij}(\bar{x})$, $b_i(\bar{x})$, $h(\bar{x})$, $c(\bar{x})$ and $f(\bar{x})$ are continuous functions of $\bar{x} = (x_1, x_2, \dots, x_n)$ in \bar{G} , and $c(\bar{x}) \leq 0$ in \bar{G} . Furthermore, $L[u]$ is uniformly elliptic in \bar{G} , i.e. there exists a positive constant m such that

$$(2) \quad \sum_{i,j=1}^n a_{ij}(\bar{x}) \xi_i \xi_j \geq m \sum_{i=1}^n \xi_i^2$$

for all n -tuples of real numbers (ξ_1, \dots, ξ_n) and all $\bar{x} \in \bar{G}$. By normalization, $a_{nn}(\bar{x}) = 1$.

In this paper we shall formulate some boundary value problems of (1) in \bar{G} . Since the coefficient of $\partial u / \partial x_n$ goes to infinity as $x_n \rightarrow 0$, it is obvious

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that the behavior of $h(\bar{x})$ near $x_n=0$ plays an important part in our formulation. A well-known example of (1) is given by the generalized axially symmetric potential equation

$$(3) \quad \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} + \frac{k}{x_n} \frac{\partial u}{\partial x_n} = 0$$

where k is a real number.

For $k < 1$, P. Brousse and H. Poncin have shown that the Dirichlet problem can be solved if smooth boundary is admitted [1]. However, for $k \geq 1$, the Dirichlet problem is in general not solvable. A. Huber has shown that there exists a unique function u in \bar{G} , which satisfies (3) and assumes bounded continuous data on S_2 ([3], [4]). Accordingly, we shall formulate our boundary value problems, based on the assumption whether $h(x_1, \dots, x_{n-1}, 0) < 1$ or $h(x_1, \dots, x_{n-1}, 0) \geq 1$.

THEOREM 1. *Let $h(\bar{x})$ be a C^2 -function in \bar{G} , even in x_n , and $h(x_1, x_2, \dots, x_{n-1}, 0) \geq 1$. Then there is at most one solution $u(\bar{x})$ of (1) which is regular in G , remains bounded when $x_n \rightarrow 0$, and assumes given continuous values $\Phi(\bar{x})$ on the boundary S_2 .*

PROOF. It is sufficient to prove that any bounded and regular solution $u(\bar{x})$ of $L[u]=0$ which vanishes on S_2 must vanish identically in G . We want to construct a barrier function $w(\bar{x})$ with the following properties: (a) $w(\bar{x})$ is positive in \bar{G} , (b) $w(\bar{x})$ converges uniformly to infinity when $x_n \rightarrow 0$ and (c) $L[w] < 0$.

If such a function $w(\bar{x})$ exists, then, for any $\varepsilon > 0$, we form $v(\bar{x}) = \varepsilon w(\bar{x}) + u(\bar{x})$. Let $P \in G$. Then, from property (b) of $w(\bar{x})$, there exists an η such that

$$v(\bar{x}) \geq 0 \quad \text{for } 0 < x_n \leq \eta.$$

We require η is so small that $P \in G_\eta = G \cap D_\eta$ where $D_\eta = \{\bar{x} | x_n > \eta\}$. This implies $v(\bar{x}) \geq 0$ on ∂G_η , and, from property (c) of $w(\bar{x})$, $L[v] < 0$. Hence, by maximum principle for elliptic operators [2], $v(\bar{x}) \geq 0$ in G_η or $u(P) \geq -\varepsilon w(P)$. Similarly, by defining $v(\bar{x}) = \varepsilon w(\bar{x}) - u(\bar{x})$, we obtain $u(P) \leq \varepsilon w(P)$. Hence $|u(P)| \leq \varepsilon w(P)$. Since ε is arbitrary, $u(P) = 0$.

The function $w(\bar{x})$ can be defined as

$$(4) \quad w(\bar{x}) = -\ln x_n - (x_1 - \alpha)^\beta + k$$

where α is chosen such that $x_1 - \alpha > 1$ for all $\bar{x} = (x_1, \dots, x_n) \in G$, whereas β and k are positive constants to be determined later.

Then

$$(5) \quad \begin{aligned} L[w] = & -a_{11}(\bar{x})\beta(\beta - 1)(x_1 - \alpha)^{\beta-2} - b_1(\bar{x})\beta(x_1 - \alpha)^{\beta-1} \\ & + (1 - h(\bar{x}))/x_n^2 + c(\bar{x})w(\bar{x}). \end{aligned}$$

Applying Taylor's theorem with remainder to the function $F(x_n) = h(\xi_1, \dots, \xi_{n-1}, x_n)$ where $\xi_i, i=1, \dots, n-1$, are considered as parameters and $(\xi_1, \dots, \xi_n, x_n) \in G$, we obtain

$$F(x_n) = F(0) + (F'(0)/1!)x_n + (F''(\xi_n)/2!)x_n^2$$

where $0 < \xi_n < x_n$.

Then, from the hypothesis on $h(\bar{x})$, we have

$$\begin{aligned} & \frac{1 - h(\xi_1, \dots, \xi_{n-1}, x_n)}{x_n^2} \\ (6) \quad &= \frac{1 - h(\xi_1, \dots, \xi_{n-1}, 0) - h(\xi_1, \dots, \xi_{n-1}, 0)}{x_n^2} \\ & \quad - \frac{h_{x_n x_n}(\xi_1, \dots, \xi_{n-1}, \xi_n)}{x_n^2} \\ & < -h_{x_n x_n}(\xi_1, \dots, \xi_{n-1}, \xi_n) \leq A \end{aligned}$$

where A is chosen to be a positive constant greater than $\max_{\bar{x} \in \bar{G}} |h_{x_n x_n}(\bar{x})|$.

Moreover,

$$(7) \quad a_{11}(\bar{x}) > m \text{ by (2).}$$

Hence, combining (5), (6) and (7), we have

$$L[w] < -m\beta(\beta - 1)(x_1 - \alpha)^{\beta-2} - \beta b_1(\bar{x})(x_1 - \alpha)^{\beta-1} + A + C(\bar{x})w(\bar{x}).$$

Choose β such that $m(\beta - 1) > \max(3B, 2)$ and $\beta(\beta - 1) > 3A/m$ where $B = \max_{\bar{x} \in \bar{G}} |b_1(\bar{x})(x_1 - \alpha)|$.

The number k is now chosen such that the equation $w(\bar{x})$ is positive everywhere in \bar{G} . Then,

$$\begin{aligned} L[w] &< -m\beta(\beta - 1) + \frac{1}{3}m\beta(\beta - 1) + A + c(\bar{x})w(\bar{x}) \\ &< -(2m\beta/3)(\beta - 1) + A < -m\beta(\beta - 1)/3 < 0. \end{aligned}$$

In order to show that the Dirichlet problem is solvable when $h(x_1, \dots, x_{n-1}, 0) < 1$, we plan to apply the Schauder theorem on the regular elliptic differential equations.

For $0 < \alpha < 1$, let $C^{m+\alpha}(\mathcal{D})$ be the set of functions $u \in C^m(\mathcal{D})$ whose m th order derivatives satisfy a Hölder condition in (\mathcal{D}) with exponent α . For $u \in C^{m+\alpha}(\mathcal{D})$, define

$$\|u\|_{m+\alpha}^{\mathcal{D}} = \max_{x \in \bar{\mathcal{D}}} \sum_{k=0}^m |\mathcal{D}^k u| + \max_{P, Q \in \bar{\mathcal{D}}} \frac{|D^m u(P) - D^m u(Q)|}{|P - Q|^\alpha}.$$

Then the Schauder result can be stated as follows. (See [5], [6].)

LEMMA (SCHAUDER). Let \mathcal{D} be a bounded domain with boundary $\partial\mathcal{D}$ of class $C^{2+\alpha}$. Let

$$(8) \quad \mathcal{L}(u) = \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} + cu = f$$

be a uniformly elliptic differential equation in \mathcal{D} . Assume that the coefficients a_{ij}, b_i, c, f are in $C^\alpha(\mathcal{D})$ satisfying $|a_{ij}|, |b_i|, |c|, |f| \leq K$, a constant and $c \leq 0$ in \mathcal{D} .

If ϕ is any continuous function on $\partial\mathcal{D}$, there exists a unique solution $u \in C^{2+\alpha}(\mathcal{D}) \cap C^0(\mathcal{D})$ of (8) with $u = \phi$ on $\partial\mathcal{D}$.

Accordingly, we formulate the following existence theorem:

THEOREM 2. Let G be a bounded domain in $x_n > 0$ with boundary $\partial G = S_1 \cup S_2$ of class $C^{2+\alpha}$ where S_1 is a subset of $x_n = 0$ and S_2 in $x_n > 0$. Let the coefficients a_{ij}, k_i, h, c and f be in $C^\alpha(\bar{G})$ satisfying $|a_{ij}|, |b_i|, |h|, |f| \leq K$, and $C \leq 0$ in \bar{G} .

If ϕ is any continuous function on ∂G , there exists a unique solution $u \in C^{2+\alpha}(G) \cap C^0(\bar{G})$ of (1) with $u = \phi$ on ∂G .

PROOF. The uniqueness of the solution can be handled easily by Hopf's maximum principle.

Extend the function ϕ continuously to the closed domain \bar{G} and denote $M = \max_{\bar{x} \in \bar{G}} \phi(\bar{x})$. Let $\eta > 0$, and denote $G_\eta = G \cap E\{x_n > \eta\}$. According to Schauder's lemma, there exists a solution $u_\eta \in C^{2+\alpha}(G_\eta) \cap C^0(\bar{G}_\eta)$ of (1) with $u_\eta = \phi$ on ∂G_η . Extend the function u_η continuously on the whole domain \bar{G} by defining $u_\eta = \phi$ on $\bar{G} \setminus \bar{G}_\eta$. In this way we define a family U of functions $\{u_\eta(\bar{x})\}$, in \bar{G} corresponding to all values of η , $0 < \eta \leq \max\{x_n | \bar{x} = (x_1 \cdots x_n) \in \bar{G}\}$.

Let η^* be a fixed, but arbitrary small number. The family U is uniformly bounded by M , and from Schauder's estimate

$$(9) \quad \|u\|_{\partial+\alpha}^{G_\eta} \leq \tilde{K}(\|f\|_\alpha^{G_\eta} + \|u\|_0^{G_\eta}) \quad ([5], [6])$$

this family is also equicontinuous in G_η . Then, by Arzela's lemma, it is possible to select a sequence $\{U_i^{(1)}(x)\} = \{u_{\eta_i}(\bar{x})\}$ so that $U_i^{(1)}$, together with its first and second derivatives, converges uniformly in G_η . The sequence $\{U_i^{(1)}(x)\}$ is chosen in the order that $i > j$ if $\eta_i < \eta_j$. Hence, for $\eta_i < \eta$, all $\{U_i^{(1)}(\bar{x})\}$ are solutions of (1) in G_{η^*} .

For each j , let $\eta_j^* = \eta^*/j$. Since $\{U_i^{(1)}(\bar{x})\}$ is also a family of uniform boundedness and equicontinuity in $G_{\eta_j^*}$, we can select a uniformly convergent subsequence $\{U_i^{(2)}(\bar{x})\}$ in $G_{\eta_j^*}$ and so forth.

Now, we choose from the sequences

$$\begin{aligned} &U_1^{(1)}, U_2^{(1)}, \dots, U_i^{(1)}, \dots, \\ &U_1^{(2)}, U_2^{(2)}, \dots, U_i^{(2)}, \dots \end{aligned}$$

the “diagonal” subsequence $\{U_n^{(n)}(\bar{x})\}$ which converges uniformly to a regular solution $u(\bar{x})$ of (1) in G and assumes $\phi(\bar{x})$ on S_2 . We still have to prove that for any $Q \in S_1$, $u(\bar{x})$ is continuous and $u(Q) = \phi(Q)$.

Let $Q = \bar{x}^0 = (x_1^0, x_2^0, \dots, x_{n-1}^0, 0) \in S_1$. We want to construct a barrier function $v(\bar{x})$ with the following properties (a) it is continuous in a sufficiently small neighborhood $w_Q = \{\bar{x} \mid |\bar{x} - \bar{x}^0| < \rho, x_n \geq 0\}$, (b) it vanishes at Q , (c) it is positive in $w_Q \setminus Q$ and (d) $L[v] < -1$.

The function $v(\bar{x})$ can be defined as

$$(10) \quad v(x_1, \dots, x_{n-1}, x_n) = x_n^\beta + \sum_{i=1}^n (x_i - x_i^0)^2$$

where $0 < \beta < 1$.

It is clear that $v(\bar{x})$ satisfies properties (a), (b) and (c). However, we still have to determine the exact value of β such that $L[v] < -1$.

By elementary computations,

$$\begin{aligned} L[v] &= 2(a_{11}(\bar{x}) + \dots + a_{n-1, n-1}(\bar{x})) + \beta(\beta - 1)x_n^{\beta-2} \\ &\quad + 2[b_1(\bar{x})(x_1 - x_1^0) + \dots + h_{n-1}(\bar{x})(x_{n-1} - x_{n-1}^0)] \\ &\quad + \beta h(\bar{x})x_n^{\beta-2} + c(\bar{x})v(\bar{x}) \\ &< 2(n - 1)K + 2\rho(n - 1)K + \beta[\beta - 1 + h(\bar{x})]x_n^{\beta-2}. \end{aligned}$$

Choose β such that $0 < \beta < 1 - h(x_1, \dots, x_{n-1}, 0)$. Then, by continuity, and for sufficiently small values of ρ , we have $L[v] < -1$.

Since $\phi(\bar{x})$ is continuous, for any $\varepsilon > 0$, we can find a semicircular neighborhood $w'_Q \subset w_Q$ such that

$$\phi(Q) - \varepsilon \leq \phi(P) \leq \phi(Q) + \varepsilon.$$

Consider the functions

$$A(P) = \phi(Q) - \varepsilon - kv(P) \quad \text{and} \quad B(P) = \phi(Q) + \varepsilon + kv(P).$$

For a sufficiently large positive number k , we have $B(P) > M > A(P)$ on the semicircular part of the boundary w'_Q and $L[A] > K$ and $L[B] < -K$ everywhere in w'_Q . Choose a nonempty domain $w_q = w'_Q \cap E\{x_n > q > 0\}$. On ∂w_q , $B(P) > M \geq U_n^n(P)$, where U_n^n is an arbitrary member of the sequence $\{U_n^n\}$ and satisfies (1) in w_q , whereas in w_q , we have $L[B - U_n^n] = L[B] - L[U_n^n] < -K + f(\bar{x}) \leq 0$. Then, by maximum principle, $U_n^n(P) \leq B(P)$ in w_q . Similarly, $U_n^n(P) \geq A(P)$ in w_q . Thus we have $A(P) \leq U_n^n(P) \leq B(P)$. Let $n \rightarrow \infty$;

$$A(P) \leq u(P) \leq B(P);$$

and then let $P \rightarrow Q$;

$$f(Q) - \varepsilon \leq \lim_{P \rightarrow Q} u(P) \leq f(Q) + \varepsilon$$

or

$$\lim_{P \rightarrow Q} u(P) = f(Q).$$

REMARK. M. Schechter [5] has established an existence theorem of Dirichlet problem of $Lu=f$, similar to Theorem 2 by means of Schauder's lemma, but the conditions on the coefficients of the equation are different from ours and his proof does not make use of the construction of barrier function.

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