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A TWO-COLOR THEOREM FOR ANALYTIC MAPS IN R^n

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ABSTRACT. If $f_1, \cdots, f_k$ are real analytic functions on R^n then the connected components of $R^n \backslash [f_1^{-1}(0) \cup \cdots \cup f_k^{-1}(0)]$ can be "colored" with two colors so that two components will have different colors whenever their common boundary contains a topological $(n-1)$-manifold.

If $f_1, \cdots, f_k$ are real analytic functions on R^n and if C is a connected component of $R^n \backslash [f_1^{-1}(0) \cup \cdots \cup f_k^{-1}(0)]$ then we shall say that C is a *country* of the *analytic map in* R^n *determined by* $f_1, \cdots, f_k$. The *border* of C is its topological boundary ∂C. An *admissible coloring* of a map in R^n is a function whose domain is the set of countries and which assigns different values (colors) to contiguous countries—two countries C, D being *contiguous* if and only if $C \neq D$ and their common border $\partial C \cap \partial D$ contains a topological $(n-1)$-manifold. Our main result is:

THEOREM. *Every analytic map in* R^n *can be colored admissibly with two colors.*

It is popularly known that in R^2 some maps require four colors and that five will always suffice [6]. In higher dimensions infinitely many colors may be required for an admissible coloring. For example, consider the map where each country is constructed by connecting a slab perpendicular to the X-axis in the half-space $z>0$ to a slab perpendicular to the Y-axis in the half-space $z<0$. More exactly, let a point $(x, y, z, \cdots) \in R^n$ lie in the kth country C_k if and only if either (1) $z>0$ and $k<x<k+1$, or (2) $z=0$ and $k<x$, $y<k+1$, or (3) $z<0$ and $k<y<k+1$. Then each country is contiguous with every other one since $(\partial C_i) \cap (\partial C_j)$ contains the $(n-1)$-manifold $\{(x, y, z, \cdots) \in R^n | z=0, i<x<i+1, j<y<j+1\}$. Furthermore, H. Tietze [7] and A. S. Besicovitch [1] have shown that an infinite number of colors may be required even if the countries of the map are all convex bodies lying in R^3.

For each $x \in R^n$ let $\mathcal{O}_x$ denote the ring of germs of real analytic functions at x. We denote the germ of the function f at x by f_x and we also interpret

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"$g_x \in \mathcal{O}_x$" to mean that g is a real analytic function defined in some open neighborhood of x. The interior of the domain of a function f will be denoted $D[f]$. Each ring $\mathcal{O}_x$ is a unique factorization domain (cf. [**4**, p. 72]) and when we refer to $uq_1^{e_1} \cdots q_k^{e_k}$ as "the factorization" of $p \in \mathcal{O}_x$ we mean that u (if it appears) is a unit in $\mathcal{O}_x$, the q_i are nonassociate irreducibles in $\mathcal{O}_x$, every integer e_i is positive, and $p = uq_1^{e_1} \cdots q_k^{e_k}$. (Of course k may be 0.) The e_i's will be called *the exponents of the factorization* of p.

For certain real analytic functions the variety of singular points cannot be very large, as the following proposition shows.

PROPOSITION 1. *Let $f \neq 0$ be a real analytic function on $\boldsymbol{R}^n$ such that for every $x \in f^{-1}(0)$ all the exponents of the factorization of f_x are* 1. *Then no topological $(n-1)$-manifold is contained in the set of singuar points of f.*

PROOF. Suppose, on the contrary, that $M \neq \varnothing$ is an $(n-1)$-manifold on which f and all of its first partial derivatives vanish. Since $f \neq 0$ some partial derivative of f of some order must be nonzero on part of M. Thus we can select g, a partial derivative of f, and $w = (w_1, \cdots, w_n) \in M$ so that $g|_M = 0$ but $((\partial/\partial y)g)(w) \neq 0$, where y is one of the x_i's, say x_n. Now let x denote $(x_1, \cdots, x_{n-1})$ so that (x, y) denotes $(x_1, \cdots, x_n)$. The Implicit Function Theorem shows that there exists a neighborhood U of w such that the zeros of $g(x, y)$ which lie in U are exactly the graph G of some continuous function $y = \phi(x)$ which is defined in some neighborhood of $(w_1, \cdots, w_{n-1}) \in \boldsymbol{R}^{n-1}$. Thus $M \cap U$ is contained in G since $g|_M = 0$. Moreover $M \cap U$ is a nonempty (it contains w) open subset of the manifold M and so it is locally homeomorphic to $\boldsymbol{R}^{n-1}$. The mapping $\pi: (x, y) \in G \to x \in \boldsymbol{R}^{n-1}$ is a homeomorphism because ϕ is continuous. Thus $V = \pi(M \cap U) \subseteq \boldsymbol{R}^{n-1}$ is locally homeomorphic to $\boldsymbol{R}^{n-1}$ and by Invariance of Domain [**3**] it must be a nonempty open subset of $\boldsymbol{R}^{n-1}$. For $x \in V$ we know that $f(x, \phi(x)) = (\partial f/\partial y)(x, \phi(x)) = g(x, \phi(x)) = 0$ because $(x, \phi(x)) \in M$.

If we assume that f_w and g_w are relatively prime then by a corollary of the Weierstrass Preparation Theorem (cf. [**4**, p. 90]) there exist $a_w, b_w \in \mathcal{O}_w$ such that $r_w = a_w f_w + b_w g_w$ is nonzero and the function r depends only on x. Now

$$0 = a(x, \phi(x))f(x, \phi(x)) + b(x, \phi(x))g(x, \phi(x)) = r(x)$$

for every $x \in V' = V \cap D[a(\cdot, \phi(\cdot))] \cap D[b(\cdot, \phi(\cdot))]$. But V' is an open neighborhood of $(w_1, \cdots, w_{n-1}) \in \boldsymbol{R}^{n-1}$, so $r_w = 0$. This contradiction shows that f_w and g_w have an irreducible common divisor. Since w is not a singular point of g, g_w is irreducible, and so g_w must divide f_w, viz. $f_w = g_w h_w$ for some $h_w \in \mathcal{O}_w$. Our hypothesis about the factorization of each f_x when $x \in f^{-1}(0)$ implies that g_w and h_w are relatively prime. Thus there exist $c_w, d_w \in \mathcal{O}_w$ such that $s_w = c_w g_w + d_w h_w$ is nonzero and the function s

depends only on x. Since $g(x, \phi(x))=0$ for $x \in V$, we have $s(x)=d(x, \phi(x))h(x, \phi(x))$ for every x in

$$V'' = V \cap D[c(\cdot, \phi(\cdot))] \cap D[d(\cdot, \phi(\cdot))] \cap D[h(\cdot, \phi(\cdot))].$$

Since $V'' \subseteq \boldsymbol{R}^{n-1}$ is an open neighborhood of $(w_1, \cdots, w_{n-1})$, to contradict $s_w \neq 0$ it suffices to find an open set $W \subseteq \boldsymbol{R}^{n-1}$ which contains $(w_1, \cdots, w_{n-1})$ and on which $h(x, \phi(x))$ vanishes. Let

$$W = \{x \in V \mid (\partial g/\partial y)(x, \phi(x)) \neq 0\} \cap D[h, (\cdot, \phi(\cdot))].$$

Since $\partial f/\partial y = (\partial g/\partial y)h + g\, \partial h/\partial y$, for each $x \in W$ we have $0=(\partial f/\partial y)(x, \phi(x))=(\partial g/\partial y)(x, \phi(x))h(x, \phi(x))$. Thus for each $x \in V'' \cap W$ we have $s(x)=d(x, \phi(x))h(x, \phi(x))=0$ and so $s_w=0$.

Now we establish a special case of our theorem.

PROPOSITION 2. *Let $f \neq 0$ be a real analytic function on $\boldsymbol{R}^n$ such that for every $x \in f^{-1}(0)$ all the exponents of the factorization of f_x are* 1. *Then* $\gamma=\operatorname{sgn}(f)$ *induces an admissible coloring of the analytic map determined by f by two colors.*

PROOF. Since γ is constant on each country and since $\gamma(x)=1$ or -1 when $x \in \boldsymbol{R}^n \backslash f^{-1}(0)$, it suffices to show that if C and D are two contiguous countries then f is positive in one and negative in the other. Let $M \subset (\partial C \cap \partial D)$ be an $(n-1)$-manifold and, using the first proposition, pick a regular point $w=(w_1, \cdots, w_n) \in M$.

First we use the Implicit Function Theorem to show that w lies in the interior of $\bar{C} \cup \bar{D}$. At least one of the numbers $(\partial f/\partial x_i)(w)$ must be nonzero and we adjust our notation so that this occurs when $i=n$. We also set $y=x_n$ and $x=(x_1, \cdots, x_{n-1})$. Then there exists a continuous function ϕ, whose domain V is a connected open neighborhood of $(w_1, \cdots, w_{n-1})$ and whose range $\phi(V)$ lies in some interval (a, b) containing w_n such that the zeros of f which lie in $U=V \times (a, b)$ are exactly the graph G of ϕ: $V \to \boldsymbol{R}$. Since $U_+=\{(x, y) \in U \mid y>\phi(x)\}$ and $U_-=\{(x, y) \in U \mid y<\phi(x)\}$ are disjoint connected open sets which miss $f^{-1}(0)$ and since $U=U_+ \cup G \cup U_-$ is a neighborhood of w, C must meet, and hence contain, either U_+ or U_-. Likewise D must contain the other. Thus $w \in U \subseteq \operatorname{interior}(\bar{C} \cup \bar{D})$.

Were f positive (or negative) throughout both C and D then f does not change sign in U. Then $f(w)=0$ is a minimum (or maximum) for f in U and so $(\partial f/\partial x_i)(w)=0$ for $i=1, \cdots, n$. Since w is not a singular point this is a contradiction and the proposition is established.

Let $\mathcal{O}^c$ denote the sheaf over $\boldsymbol{C}^n$ of germs of holomorphic functions. Then for each $z \in \boldsymbol{C}^n$ the ring $\mathcal{O}^c_z$ is a unique factorization domain (cf. [**4**, p. 72]). We can, and frequently shall, consider $\mathcal{O}_x$ to be a subset of $\mathcal{O}^c_x$ when $x \in \boldsymbol{R}^n$ because the convergent power series which determines a germ in $\mathcal{O}_x$ also determines a germ in $\mathcal{O}_x^c$. If $p=g_z \in \mathcal{O}_z^c$ then $\bar{p}$ will denote the germ in $\mathcal{O}_{\bar{z}}^c$ of the map $w \to (g(\bar{w}))^-$. Note that if $x \in \boldsymbol{R}^n$ and $p=g_x \in \mathcal{O}_x^c$ then $p\bar{p}$ may be viewed as an element of $\mathcal{O}_x$ because $g(y)(g(\bar{y}))^- \in \boldsymbol{R}$ for every $y \in \boldsymbol{R}^n \cap D[g]$.

LEMMA. *If p is irreducible in $\mathcal{O}_x$ but reducible in $\mathcal{O}_x^c$ then its factorization in $\mathcal{O}_x^c$ has the form $p=uq\bar{q}$, where u is a unit in $\mathcal{O}_x$ and q and $\bar{q}$ are nonassociate irreducibles in $\mathcal{O}_x^c$.*

PROOF. If $q \in \mathcal{O}_x^c$ is an irreducible which divides p then $\bar{q}$ is an irreducible which divides $\bar{p}=p$. So $q\bar{q} \in \mathcal{O}_x$ and divides p^2. Since p is reducible in $\mathcal{O}_x^c$, p^2 and $q\bar{q}$ cannot be associates. Thus p and $q\bar{q}$ must be associates in $\mathcal{O}_x$, viz. $p=uq\bar{q}$ for some unit $u \in \mathcal{O}_x$. If q and $\bar{q}$ are associates then there exists a unit $v \in \mathcal{O}_x^c$ such that $p=uvq^2=((uv)^{1/2}q)^2$. Let $(uv)^{1/2}q=g_x \in \mathcal{O}_x^c$. Since $p=(g_x)^2$, the function g^2 must be real-valued on the open set $\boldsymbol{R}^n \cap D[g]$. This implies that one of the analytic functions g, ig must be real-valued throughout $\boldsymbol{R}^n \cap D[g]$. Thus either g_x or $(ig)_x$ is an irreducible element of $\mathcal{O}_x$ which divides p. But p is irreducible in $\mathcal{O}_x$ and yet $p=(g_x)^2=-(ig)_x^2$. This contradiction shows that q and $\bar{q}$ are in fact nonassociate.

COROLLARY. *The germs $p, q \in \mathcal{O}_x$ are relatively prime if and only if they are relatively prime in $\mathcal{O}_x^c$.*

Let $\mathcal{O}^*$ denote the sheaf of germs of nowhere vanishing real analytic functions on $\boldsymbol{R}^n$ and let $\mathcal{M}^*$ denote the sheaf of germs $(f/g)_x$ where $f_x, g_x \in \mathcal{O}_x \backslash \{0\}$ and $x \in \boldsymbol{R}^n$. Then $\mathcal{O}^*$ may be viewed as a subsheaf of $\mathcal{M}^*$ and the quotient sheaf $\mathcal{D}=\mathcal{M}^*/\mathcal{O}^*$ is the sheaf of real analytic divisors. Letting τ denote the quotient map we obtain the exact sequence

$$1 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{M}^* \xrightarrow{\tau} \mathcal{D} \longrightarrow 1. \tag{a}$$

The germ of the set $V \subseteq \boldsymbol{C}^n$ at $z \in \boldsymbol{C}^n$ will be denoted V_z, the ideal of that germ, namely $\{f_z \in \mathcal{O}_z^c | f_z \text{ vanishes on } V_z\}$, will be denoted $\mathrm{Id}(V_z)$, and for each ideal $I \subseteq \mathcal{O}_z^c$, $\mathrm{Rad}(I)=\{p \in \mathcal{O}_z^c | p^n \in I \text{ for some integer } n>0\}$ is the *radical* of I (cf. [4]).

PROPOSITION 3. *If f is a real analytic function on $\boldsymbol{R}^n$ then there exists a real analytic function g on $\boldsymbol{R}^n$ such that whenever $x \in \boldsymbol{R}^n$ and $up_1^{e_1} \cdots p_k^{e_k}$ is the factorization of f_x in $\mathcal{O}_x$ then $g_x=vp_1 \cdots p_k$ for some unit $v \in \mathcal{O}_x$.*

PROOF. Let $\delta: \boldsymbol{R}^n \to \mathcal{D}$ be defined by $\delta(x)=p_1 \cdots p_k\mathcal{O}_x^* \in \mathcal{D}_x$, where k and the p_i's depend on x as described above. The first step is to prove that δ is a divisor, i.e. that it is continuous. We shall do this by showing that for each $x \in \boldsymbol{R}^n$ there exists a real analytic function h defined in some neighborhood U of x such that $\delta(y)=h_y\mathcal{O}_y^*$ for every $y \in U$. In fact, we take h

to be any real analytic function such that $\delta(x)=h_x\mathcal{O}_x^*$, e.g. pick h so that $h_x=p_1\cdots p_k$.

To see that this choice works we shift our attention from $\mathcal{O}_x$ to $\mathcal{O}_x^c$. If $p_1,\cdots,p_j$ are those of the p_i's which are reducible in $\mathcal{O}_x^c$ then the lemma shows that

$$f_x = wq_1^{e_1}\bar{q}_1^{e_1}\cdots q_j^{e_j}\bar{q}_j^{e_j}p_{j+1}^{e_{j+1}}\cdots p_k^{e_k} \tag{b}$$

for some unit $w\in\mathcal{O}_x^c$. Since the corollary shows that $q_1,\bar{q}_1,\cdots,p_k$ are nonassociate, (b) must be the unique factorization of f_x in $\mathcal{O}_x^c$. Thus $\mathrm{Rad}(f_x)=(q_1\bar{q}_1\cdots p_k)=(p_1\cdots p_k)=(h_x)$. Applying the Nullstellensatz gives $(h_x)=\mathrm{Rad}(f_x)=\mathrm{Id}([f^{-1}(0)]_x)$, and since $\mathrm{Id}(f^{-1}(0))$ is a coherent sheaf of ideals there exists an open subset W of $\boldsymbol{C}^n$ which contains x such that $(h_z)=\mathrm{Id}([f^{-1}(0)]_z)$ for every $z\in W$ (cf. [**4**, pp. 138–141]). Let $U=W\cap\boldsymbol{R}^n$, pick $y\in U$, let $r_1^{d_1}\cdots r_m^{d_m}$ be the factorization of f_y in $\mathcal{O}_y$, and take k to be a real analytic function such that $k_y=r_1\cdots r_m$. Then $\delta(y)=k_y\mathcal{O}_y^*$, and applying the Nullstellensatz to k as we did to h gives $(k_y)=\mathrm{Id}([f^{-1}(0)]_y)$. But $(h_y)=\mathrm{Id}([f^{-1}(0)]_y)$ also, and so k_y and h_y are associates. Hence $\delta(y)=h_y\mathcal{O}_y^*$, and so this formula must hold for every $y\in U$. Thus δ is a divisor.

The second part of our proof is in the spirit of Cousin's Second Problem. We show that every divisor, and δ in particular, is principal. From (a) we obtain the exact sequence

$$H^0(\boldsymbol{R}^n,\mathcal{M}^*)\xrightarrow{\tau_*}H^0(\boldsymbol{R}^n,\mathcal{D})\longrightarrow H^1(\boldsymbol{R}^n,\mathcal{O}^*).$$

Since all we wish to know is that τ_* is surjective, we need only prove that $H^1(\boldsymbol{R}^n,\mathcal{O}^*)=1$. From the exact sequence

$$0\longrightarrow\mathcal{O}\xrightarrow{\exp}\mathcal{O}^*\longrightarrow\boldsymbol{Z}_2\longrightarrow 0$$

we obtain the exact sequence

$$H^1(\boldsymbol{R}^n,\mathcal{O})\to H^1(\boldsymbol{R}^n,\mathcal{O}^*)\to H^1(\boldsymbol{R}^n,\boldsymbol{Z}_2).$$

Since $H^1(\boldsymbol{R}^n,\boldsymbol{Z}_2)=0$ and $H^1(\boldsymbol{R}^n,\mathcal{O})=0$ (cf. [**2**, p. 87]), we have $H^1(\boldsymbol{R}^n,\mathcal{O}^*)=1$.

The theorem can now be proved quite simply.

Proof of the Theorem. If $f_1,\cdots,f_k$ are the real analytic functions which determine the map then the function $f=f_1\cdots f_k$ will determine the same map, and so will the function g which can be obtained by applying Proposition 3 to f. Since Proposition 2 can be applied to this g the proof is complete.

Remarks. (1) No similar two-color or three-color theorem holds in $\boldsymbol{R}^2$ for C^∞ curves. For example if f is a C^∞ function defined on $\boldsymbol{R}$ such that $f(x)>0$ if $|x|<1$ and $f(x)=0$ if $|x|\geqq 1$ then the curves $y=f(x)$, $y=0$, and $y=-f(x+1)$ determine a map which requires four colors.

(2) Thinking of countries as vertices and thinking of contiguity as giving edges between vertices will of course give our theorem a graph theoretic interpretation. Some results about coloring the vertices of a graph with two colors can be found in [**6**].

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