PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 2, July 1973

ON THE BOUNDEDNESS AND UNBOUNDEDNESS OF CERTAIN CONVOLUTION OPERATORS ON NILPOTENT LIE GROUPS

ROE GOODMAN

ABSTRACT. One method of proving irreducibility of the "principal series" representations of semisimple Lie groups involves showing that a certain nonintegrable function on a nilpotent subgroup X cannot be regularized to give a bounded convolution operator on $L_2(X)$. This note gives an elementary proof of this unboundedness property for the groups X which occur in real-rank one semisimple groups.

Introduction. Let X be a connected, simply-connected nilpotent Lie group with a one-parameter group $\{\delta_r; 0<r<\infty\}$ of *dilations* and a *norm function* $|x|$, as defined by Knapp and Stein in [**2**]. For example, $X=\boldsymbol{R}^n$ with $\delta_r x=rx$ (scalar multiplication), and $|x|=\|x\|^n$, where $\|\cdot\|$ is the Euclidean norm. Our purpose in this note is to obtain some elementary results on the boundedness and unboundedness of regularizations of the nonintegrable function $|x|^{-1}$, acting by convolution on various function spaces. In particular, we obtain an independent proof of the "unboundedness theorem" of [**2**], i.e. the fact that no distribution obtained by regularization of $|x|^{-1}$ at the identity is a bounded convolution operator on $L_2(X)$. Our approach (suggested by the proof of the discrete version of this theorem in Hardy-Littlewood-Pólya [**1**, p. 214]) is to exploit the nonintegrability of $|x|^{-1}$ at infinity. This requires control over the singularity of $|x|^{-1}$ at the identity, which we gain by proving a simple "boundedness" theorem in §1. The general unboundedness theorem of Knapp and Stein is an easy corollary of the unboundedness of the "minimal" regularization of $|x|^{-1}$, which we establish in §2.

1. **Boundedness theorem.** Before stating the theorem, we recall some properties of the norm function $|x|$ (cf. [**2**, §§2 and 5]). We have $x\to|x|$ continuous on X and C^∞ on $X-\{e\}$ (we use e to denote the identity element of X), with $|x^{-1}|=|x|$ and $|x|>0$ if $x\neq e$. Under dilations it transforms by $|\delta_r x|=r^q|x|$, where $q>0$. If dx denotes a Haar measure on X, then

Received by the editors May 16, 1972.
AMS (MOS) subject classifications (1970). Primary 22E30, 44A25.

$d(\delta_r x)=r^q\,dx$, so that $|x|^{-1}\,dx$ is invariant under $\{\delta_r\}$. The following inequalities hold:

$$|xy| \leqq A \max\{|x|, |y|\}, \tag{1}$$

$$\big|\,|xy| - |y|\,\big| \leqq C_0\,|x|^d\,|y|^{1-d} \quad \text{if } |x| \leqq |y|. \tag{2}$$

Here $A>1$, while $0<d\leqq 1$. (If $\{p_i\}$ are the eigenvalues of the generator D of $\{\delta_r\}$, then $q=\sum p_i$ and $d=\min\{p_i\}/q$.) We also need the following integration formula: If $f(x)=\phi(|x|)$, where ϕ is a nonnegative Borel function on $\boldsymbol{R}^+=(0,\infty)$, then

$$\int_X f(x)\,dx = \int_{\boldsymbol{R}^+} \phi(t)\,dt, \tag{3}$$

provided we normalize the Haar measure dx so that $\int_{1\leqq|x|\leqq e} |x|^{-1}\,dx=1$.

THEOREM 1. *Let σ be the distribution*

$$\sigma(f) = \int_{|x|\leqq 1} \{f(x) - f(e)\}\,|x|^{-1}\,dx,$$

$f\in C_0^\infty(X)$. Let $\mathscr{R}$ be the subspace of $C_0^\infty(X)$ consisting of all f of the form $f(x)=\phi(|x|)$, where $\phi\in C_0^\infty(\boldsymbol{R}^+)$. For such a function f, define

$$N(f) = \left\{\int_0^\infty |\phi'(t)|^2\,(1+t^{2-d})\,dt\right\}^{1/2} \tag{4}$$

(d as in inequality (2)*). Then there exists a constant $\beta>0$ so that*

$$\|\sigma * f\|_2 \leqq \beta N(f) \tag{5}$$

for all $f\in\mathscr{R}$.

PROOF. Define $(L_x f)(y)=f(x^{-1}y)$. Then by Schwarz' inequality,

$$\|\sigma * f\|_2^2 \leqq C_\varepsilon \int_{|x|\leqq 1} \|L_x f - f\|_2^2 \frac{dx}{|x|^{1+\varepsilon}}, \tag{6}$$

where $C_\varepsilon=\int_{|x|\leqq 1} |x|^{-1+\varepsilon}\,dx$, and $\varepsilon>0$. Suppose that $f\in\mathscr{R}$, $f(x)=\phi(|x|)$. Using the integration formula (3), to represent f as the integral of ϕ', and applying Schwarz' inequality again, we get the bound

$$|f(x^{-1}y) - f(y)|^2 \leqq \big|\,|x^{-1}y| - |y|\,\big| \int_{[|x^{-1}y|,|y|]} |\phi'(t)|^2\,dt. \tag{7}$$

We will integrate (7) to obtain an estimate for the integrand in (6). Consider first the region $|y|\geqq A|x|$ (the constant A as in (1)). In this region $[|x^{-1}y|, |y|] \subseteq [A^{-1}|y|, A|y|]$ and (2) holds. Hence

$$\int_{|y| \geqq A|x|} |f(x^{-1}y) - f(y)|^2 \, dy \leqq C_0 \, |x|^d \int_{|y| \geqq A|x|} \left\{ \int_{A^{-1}|y|}^{A|y|} |\phi'(t)|^2 \, dt \right\} |y|^{1-d} \, dy$$
$$\leqq C_1 \, |x|^d \int_0^\infty |\phi'(t)|^2 \, t^{2-d} \, dt,$$

where we have interchanged the order of integration and used formula (3) again, and the constant C_1 is independent of ϕ. In the region $|y| \leqq A|x|$ we simply use the estimate $|x^{-1}y| \leqq A^2|x|$ to obtain from (7) the bound

$$\int_{|y| \leqq A|x|} |f(x^{-1}y) - f(y)|^2 \, dy \leqq 2A^2 \, |x| \int_0^\infty |\phi'(t)|^2 \, dt.$$

Assume now that $|x| \leqq 1$. Then $|x| \leqq |x|^d$, since $d \leqq 1$, so by the estimates just made,

$$\|L_x f - f\|_2^2 \leqq C_2 \, |x|^d \int_0^\infty |\phi'(t)|^2 \, (1 + t^{2-d}) \, dt, \tag{8}$$

C_2 a constant independent of ϕ. If we take $\varepsilon = d/2$ in (6) and use estimate (8), we obtain (5). Q.E.D.

2. **Unboundedness theorem.**

THEOREM 2. *Let T be the distribution*

$$T(f) = \sigma(f) + \int_{|x| \geqq 1} f(x) \, |x|^{-1} \, dx$$

(σ as in Theorem 1). Then left convolution by T is not bounded on $L_2(X)$.

PROOF. If ν is the distribution

$$\nu(f) = \int_{|x| \geqq 1} f(x) \, |x|^{-1} \, dx,$$

then it is easy to show that convolution by ν is not bounded on L_2. Indeed, let $\{\phi_n\} \subseteq C_0^\infty(\boldsymbol{R}^+)$ be a sequence of nonnegative functions such that $\phi_n(t) = t^{-1/2}(\log t)^{-1}$ in the interval $e \leqq t \leqq A^2 e^n$, with ϕ_n smoothly cut off outside this interval in such a way that $\sup_n \int_0^\infty \phi_n(t)^2 \, dt < \infty$ (this is certainly possible, since $\int_e^\infty t^{-1}(\log t)^{-2} \, dt < \infty$). Set $f_n(x) = \phi_n(|x|)$. By formula (3), $\{f_n\}$ is a bounded sequence in the L_2 norm. On the other hand, when $e \leqq |x| \leqq e^n$, then

$$(f_n * f_n)(x) \geqq \int_{A|x| \leqq |y| \leqq Ae^n} f_n(x^{-1}y) f_n(y) \, dy$$

But in the range $A|x| \leqq |y|$, $e \leqq |x| \leqq e^n$, one has also $e \leqq A^{-1}|y| \leqq |x^{-1}y| \leqq A|y|$. Hence from the definition of f_n we get the lower bound

$$(f_n * f_n)(x) \geqq C_1 \int_{A|x| \leqq |y| \leqq Ae^n} |y|^{-1} (\log |y|)^{-2}\, dy$$
$$\geqq C_2 (\log |x|)^{-1} \quad \text{if } |x| \leqq e^{n/2},$$

where $C_2 > 0$ is independent of n.

Integrating this estimate in the range $e \leqq |x| \leqq e^{n/2}$, we conclude that $\nu(f_n * f_n) \geqq C_2 \log(n/2)$, and hence convolution by ν is not bounded on L_2. (*Remark.* $\nu \in L_p$ for all $p > 1$, so by Young's inequality convolution by ν is bounded from L_r to L_2 for any $r < 2$. The sequence $\{f_n\}$ was chosen to be bounded in L_2 but not bounded in L_r for any $r < 2$.)

To prove Theorem 2, we only need to show that a sequence $\{f_n\}$ as above can be constructed with the additional property that $\sigma(f_n * f_n)$ remains bounded as $n \to \infty$. By Theorem 1 this will be the case provided that $\sup_n N(f_n) < \infty$. This can be easily done, however. The only delicate point is to truncate the functions ϕ_n slowly enough as $t \to \infty$ so that ϕ_n' is sufficiently small (this keeps $N(f_n)$ bounded), but yet rapidly enough to keep $\|f_n\|_2$ bounded. Specifically, we construct $\phi_n \in C_0^\infty(\boldsymbol{R}^+)$ as above, $\phi_n(t) = t^{-1/2}(\log t)^{-1}$ in $e \leqq t \leqq A^2 e^n \equiv a_n$, requiring that $\phi_n(t) = 0$ for $t > (1 + e^{-nd}) a_n \equiv b_n$, and taking a fixed cut-off in $1 \leqq t \leqq e$. In the "cut-off" interval $[a_n, b_n]$ we require

$$0 \leqq \phi_n(t) \leqq M n^{-1} e^{-n/2}, \qquad |\phi_n'(t)| \leqq M \exp[n(-3/2 + d)],$$

where M is a fixed constant. This can certainly be done, since we can construct a C^1 function ψ_n with these properties (the estimate for the derivative being obtained by comparison with a linear cut-off in $[a_n, b_n]$), and then set $\phi_n = \theta * \psi_n$, where $\theta \in C_0^\infty(\boldsymbol{R})$ is fixed.

Suppose the sequence $\{\phi_n\}$ has been constructed, and set $f_n(x) = \phi_n(|x|)$. Then

$$\|f_n\|_2^2 \leqq C + \int_e^{a_n} t^{-1} (\log t)^{-2}\, dt + (b_n - a_n) M^2 n^{-2} e^{-n}.$$

But $b_n - a_n = A^2 e^{(1-d)n}$, and $d > 0$, so $\sup_n \|f_n\|_2 < \infty$. Furthermore,

$$N(f_n)^2 \leqq C + \int_e^{a_n} \frac{2(1 + t^{2-d})}{t^3 (\log t)^2}\, dt$$
$$+ (b_n - a_n)(1 + b_n^{2-d}) \sup_{a_n \leqq t \leqq b_n} |\phi_n'(t)|^2.$$

By construction of ϕ_n, the right side is bounded as $n \to \infty$, as required.

COROLLARY (KNAPP AND STEIN). *Let μ be any distribution on X such that $\mu=|x|^{-1}$ on $X\sim\{e\}$. Then convolution by μ is unbounded on $L_2(X)$.*

PROOF. Let T be the distribution of Theorem 2. Then $\mu-T$ vanishes on $X-\{e\}$, and hence is a finite linear combination of derivatives of the point mass at e. Pick a fixed, nonnegative $\phi=\phi^* \in C_0^\infty(X)$, and set $\phi_r(x)=(\log r)^{-1/2}r^{q/2}\phi(\delta_r x)$. Since $d(\delta_r x)=r^q\,dx$, while $|x|^{-1}\,dx$ is invariant under δ_r, we have

$$\begin{aligned}(\log r)T(\phi_r * \phi_r) &= \iint_{|x|\leq r^q} [\phi(xy) - \phi(y)]\phi(y)\,|x|^{-1}\,dx\,dy \\ &\quad + \iint_{|x|>r^q} \phi(y)\phi(xy)\,|x|^{-1}\,dx\,dy \\ &= T(\phi * \phi) - q(\log r)\,\|\phi\|_2^2.\end{aligned}$$

Also $\|\phi_r\|_2=(\log r)^{-1/2}\|\phi\|_2$. Hence $T(\phi_r * \phi_r)$ and $\|\phi_r\|_2$ are bounded as $r\to\infty$. On the other hand, if $\{X_i\}$ is an eigenvector for the differential of δ_e, then $(\partial/\partial X_i)\phi_r * \phi_r(e)=(\log r)^{-1}r^{p_i}(\partial\phi/\partial X_i) * \phi(e)$. Since all $p_i>0$, it is clear that $\mu(\phi_r * \phi_r)$ cannot be bounded as $r\to\infty$ unless $\mu-T$ is a multiple of the identity operator. Hence the assumption that convolution by μ is bounded on L_2 leads to the conclusion that convolution by T is bounded on L_2, contradicting Theorem 2. Q.E.D.

REFERENCES

1. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd ed., Cambridge Univ. Press, New York, 1952. MR **13**, 727.

2. A. W. Knapp and E. M. Stein, *Intertwining operators for semi-simple groups*, Ann. of Math. (2) **93** (1971), 489–578.

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903