ON THE BOUNDEDNESS AND UNBOUNDEDNESS OF CERTAIN CONVOLUTION OPERATORS ON NILPOTENT LIE GROUPS

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Abstract. One method of proving irreducibility of the "principal series" representations of semisimple Lie groups involves showing that a certain nonintegrable function on a nilpotent subgroup \( X \) cannot be regularized to give a bounded convolution operator on \( L^2(X) \). This note gives an elementary proof of this unboundedness property for the groups \( X \) which occur in real-rank one semisimple groups.

Introduction. Let \( X \) be a connected, simply-connected nilpotent Lie group with a one-parameter group \( \{ \delta_r; 0 < r < \infty \} \) of dilations and a norm function \( |x| \), as defined by Knapp and Stein in [2]. For example, \( X = \mathbb{R}^n \) with \( \delta_r x = r x \) (scalar multiplication), and \( |x| = \|x\|^n \), where \( \| \cdot \| \) is the Euclidean norm. Our purpose in this note is to obtain some elementary results on the boundedness and unboundedness of regularizations of the nonintegrable function \( |x|^{-1} \), acting by convolution on various function spaces. In particular, we obtain an independent proof of the "unboundedness theorem" of [2], i.e. the fact that no distribution obtained by regularization of \( |x|^{-1} \) at the identity is a bounded convolution operator on \( L^2(X) \). Our approach (suggested by the proof of the discrete version of this theorem in Hardy-Littlewood-Pólya [1, p. 214]) is to exploit the non-integrability of \( |x|^{-1} \) at infinity. This requires control over the singularity of \( |x|^{-1} \) at the identity, which we gain by proving a simple "boundedness" theorem in §1. The general unboundedness theorem of Knapp and Stein is an easy corollary of the unboundedness of the "minimal" regularization of \( |x|^{-1} \), which we establish in §2.

1. Boundedness theorem. Before stating the theorem, we recall some properties of the norm function \( |x| \) (cf. [2, §§2 and 5]). We have \( x \to |x| \) continuous on \( X \) and \( C^\infty \) on \( X - \{ e \} \) (we use \( e \) to denote the identity element of \( X \)), with \( |x^{-1}| = |x| \) and \( |x| > 0 \) if \( x \neq e \). Under dilations it transforms by \( |\delta_r x| = r^q |x| \), where \( q > 0 \). If \( dx \) denotes a Haar measure on \( X \), then

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$d(\delta_r x) = r^d \, dx$, so that $|x|^{-1} \, dx$ is invariant under $\{\delta_r\}$. The following inequalities hold:

(1) $|xy| \leq A \max\{|x|, |y|\}$,
(2) $| |xy| - |y| | \leq C_0 \, |x|^d \, |y|^{1-d}$ if $|x| \leq |y|$.

Here $A > 1$, while $0 < d \leq 1$. (If $\{p_i\}$ are the eigenvalues of the generator $D$ of $\{\delta_r\}$, then $q = \sum p_i$ and $d = \min\{p_i\}/q$.) We also need the following integration formula: If $f(x) = \phi(|x|)$, where $\phi$ is a nonnegative Borel function on $R^+ = (0, \infty)$, then

$$\int f(x) \, dx = \int_{R^+} \phi(t) \, dt,$$

provided we normalize the Haar measure $dx$ so that $\int_{1 \leq |x| \leq e} |x|^{-1} \, dx = 1$.

**Theorem 1.** Let $\sigma$ be the distribution

$$\sigma(f) = \int_{|x| \leq 1} \{f(x) - f(e)\} \, |x|^{-1} \, dx,$$

$f \in C_0^\infty(X)$. Let $\mathcal{R}$ be the subspace of $C_0^\infty(X)$ consisting of all $f$ of the form $f(x) = \phi(|x|)$, where $\phi \in C_0^\infty(R^+)$. For such a function $f$, define

$$N(f) = \left\{ \int_0^\infty |\phi'(t)|^2 \,(1 + t^{2-d}) \, dt \right\}^{1/2}$$

($d$ as in inequality (2)). Then there exists a constant $\beta > 0$ so that

$$\|\sigma * f\|_2 \leq \beta N(f)$$

for all $f \in \mathcal{R}$.

**Proof.** Define $(L_x f)(y) = f(x^{-1}y)$. Then by Schwarz' inequality,

$$\|\sigma * f\|_2 \leq C_{\epsilon} \int_{|x| \leq 1} \|L_x f - f\|_2^2 \frac{dx}{|x|^{1+\epsilon}} ,$$

where $C_{\epsilon} = \int_{|x| \leq 1} |x|^{-1+\epsilon} \, dx$, and $\epsilon > 0$. Suppose that $f \in \mathcal{R}$, $f(x) = \phi(|x|)$. Using the integration formula (3), to represent $f$ as the integral of $\phi'$, and applying Schwarz' inequality again, we get the bound

$$|f(x^{-1}y) - f(y)|^2 \leq | |x^{-1}y| - |y| | \int_{|y| \leq |x^{-1}y| \leq |y|} |\phi'(t)|^2 \, dt.$$

We will integrate (7) to obtain an estimate for the integrand in (6). Consider first the region $|y| \geq A|x|$ (the constant $A$ as in (1)). In this
region \([|x^{-1}y|, |y|] \subseteq [A^{-1}|y|, A|y|]\) and (2) holds. Hence
\[
\int_{|y| \geq A|x|} |f(x^{-1}y) - f(y)|^2 \, dy \leq C_0 |x|^d \int_{|y| \geq A|x|} \left\{ \int_{A^{-1}|y|}^{A|y|} |\phi'(t)|^2 \, dt \right\} |y|^{1-d} \, dy \\
\leq C_1 |x|^d \int_0^\infty |\phi'(t)|^2 t^{2-d} \, dt,
\]
where we have interchanged the order of integration and used formula (3) again, and the constant \(C_1\) is independent of \(\phi\). In the region \(|y| \leq A|x|\) we simply use the estimate \(|x^{-1}y| \leq A^2|x|\) to obtain from (7) the bound
\[
\int_{|y| \leq A|x|} |f(x^{-1}y) - f(y)|^2 \, dy \leq 2A^2 |x| \int_0^\infty |\phi'(t)|^2 \, dt.
\]
Assume now that \(|\phi| \leq 1\). Then \(|x| \leq |x|^d\), since \(d \leq 1\), so by the estimates just made,
\[
\|L_x f - f\|^2 \leq C_2 |x|^d \int_0^\infty |\phi'(t)|^2 (1 + t^{2-d}) \, dt,
\]
\(C_2\) a constant independent of \(\phi\). If we take \(\varepsilon = d/2\) in (6) and use estimate (8), we obtain (5). Q.E.D.

2. Unboundedness theorem.

THEOREM 2. Let \(T\) be the distribution
\[
T(f) = \sigma(f) + \int_{|x| \geq 1} f(x) |x|^{-1} \, dx
\]
(\(\sigma\) as in Theorem 1). Then left convolution by \(T\) is not bounded on \(L_2(X)\).

PROOF. If \(\nu\) is the distribution
\[
\nu(f) = \int_{|x| \geq 1} f(x) |x|^{-1} \, dx,
\]
then it is easy to show that convolution by \(\nu\) is not bounded on \(L_2\). Indeed, let \(\{\phi_n\} \subseteq C_0^\infty (R^+\) be a sequence of nonnegative functions such that \(\phi_n(t) = t^{-1/2}(\log t)^{-1}\) in the interval \(e \leq t \leq A^2e^n\), with \(\phi_n\) smoothly cut off outside this interval in such a way that \(\sup_n \int_0^\infty \phi_n(t)^2 \, dt < \infty\) (this is certainly possible, since \(\int_0^\infty t^{-1/2}(\log t)^{-2} \, dt < \infty\)). Set \(f_n(x) = \phi_n(|x|)\). By formula (3), \(\{f_n\}\) is a bounded sequence in the \(L_2\) norm. On the other hand, when \(e \leq |x| \leq e^n\), then
\[
(f_n * f_n)(x) \geq \int_{d|x| \leq |y| \leq A|x|} f_n(x^{-1}y) f_n(y) \, dy.
\]
But in the range $A|x| \leq |y|$, $e \leq |x| \leq e^n$, one has also $e \leq A^{-1}|y| \leq |x^{-1}y| \leq A|y|$. Hence from the definition of $f_n$ we get the lower bound

\[ (f_n * f_n)(x) \geq C_1 \int_{A|x| \leq |y| \leq A e^n} |y|^{-1} \left( \log |y| \right)^{-2} \, dy \]

\[ \geq C_2 \left( \log |x| \right)^{-1} \quad \text{if} \quad |x| \leq e^{n/2}, \]

where $C_2 > 0$ is independent of $n$.

Integrating this estimate in the range $e \leq |x| \leq e^{n/2}$, we conclude that

\[ \nu(\sigma f_n * f_n) \geq C_2 \log(n/2), \]

and hence convolution by $\nu$ is not bounded on $L_2$.

(Remark. $\nu \in L_p$ for all $p > 1$, so by Young’s inequality convolution by $\nu$ is bounded from $L_p$ to $L_q$ for any $r < 2$. The sequence $\{f_n\}$ was chosen to be bounded in $L_2$ but not bounded in $L_r$ for any $r < 2$.)

To prove Theorem 2, we only need to show that a sequence $\{f_n\}$ as above can be constructed with the additional property that $\sigma(f_n * f_n)$ remains bounded as $n \to \infty$. By Theorem 1 this will be the case provided that $\sup_N N(f_n) < \infty$. This can be easily done, however. The only delicate point is to truncate the functions $\phi_n$ slowly enough as $t \to \infty$ so that $\phi_n'$ is sufficiently small (this keeps $N(f_n)$ bounded), but yet rapidly enough to keep $\|f_n\|_2$ bounded. Specifically, we construct $\phi_n \in C_0^\infty(R^+)$ as above,

\[ \phi_n(t, r) = r^{1/2} \left( \log t \right)^{-1} \quad \text{in} \quad e = \frac{1}{A^2} e^{n} = a_n, \]

requiring that $\phi_n'(0) = 0$ for $t > (1 + e^{-nd}) a_n \equiv b_n$, and taking a fixed cut-off in $1 \leq t \leq e$. In the “cut-off” interval $[a_n, b_n]$ we require

\[ 0 \leq \phi_n(t) \leq M n^{-1} e^{-n/2}, \quad |\phi_n'(t)| \leq M \exp[n(-3/2 + d)], \]

where $M$ is a fixed constant. This can certainly be done, since we can construct a $C^1$ function $\psi_n$ with these properties (the estimate for the derivative being obtained by comparison with a linear cut-off in $[a_n, b_n]$), and then set $\phi_n = \theta * \psi_n$, where $\theta \in C_0^\infty(R)$ is fixed.

Suppose the sequence $\{\phi_n\}$ has been constructed, and set $f_n(x) = \phi_n(|x|)$. Then

\[ \|f_n\|_2^2 \leq C + \int_{e}^{a_n} t^{-1}(\log t)^{-2} \, dt + (b_n - a_n) M^2 n^{-2} e^{-n}. \]

But $b_n - a_n = A^2 e^{(1-d)n}$, and $d > 0$, so $\sup_n \|f_n\|_2 < \infty$. Furthermore,

\[ N(f_n)^2 \leq C + \int_{e}^{a_n} \frac{2(1 + t^2 - d)}{t^2(\log t)^2} \, dt \]

\[ + (b_n - a_n)(1 + b_n^{2-d}) \sup_{a_n \leq t \leq b_n} |\phi_n'(t)|^2. \]

By construction of $\phi_n$, the right side is bounded as $n \to \infty$, as required.
Corollary (Knapp and Stein). Let \( \mu \) be any distribution on \( X \) such that \( \mu = |x|^{-1} \) on \( X \sim \{ e \} \). Then convolution by \( \mu \) is unbounded on \( L_2(X) \).

Proof. Let \( T \) be the distribution of Theorem 2. Then \( \mu - T \) vanishes on \( X \sim \{ e \} \), and hence is a finite linear combination of derivatives of the point mass at \( e \). Pick a fixed, nonnegative \( \phi = \phi^* \in C_0^\infty(X) \), and set \( \phi_r(x) = (\log r)^{-1/2} r^{1/2} \phi(\delta_r x) \). Since \( d(\delta_r x) = r^d \, dx \), while \( |x|^{-1} \, dx \) is invariant under \( \delta_r \), we have

\[
(\log r) T(\phi_r * \phi) = \int_{|x| \leq r^2} \int_{|y| \leq r^2} [\phi(xy) - \phi(y)] \phi(y) |x|^{-1} \, dx \, dy \\
+ \int_{|x| > r^2} \phi(y) \phi(xy) |x|^{-1} \, dx \, dy
\]

Also \( \| \phi_r \|_2 = (\log r)^{-1/2} \| \phi \|_2 \). Hence \( T(\phi_r * \phi) \) and \( \| \phi_r \|_2 \) are bounded as \( r \to \infty \). On the other hand, if \( \{ X_i \} \) is an eigenvector for the differential of \( \delta_r \), then \( (\partial / \partial X_i) \phi_r * \phi_r(e) = (\log r)^{-1} r^{-p_i} (\partial \phi / \partial X_i) * \phi(e) \). Since all \( p_i > 0 \), it is clear that \( \mu(\phi_r * \phi_r) \) cannot be bounded as \( r \to \infty \) unless \( \mu - T \) is a multiple of the identity operator. Hence the assumption that convolution by \( \mu \) is bounded on \( L_2 \) leads to the conclusion that convolution by \( T \) is bounded on \( L_2 \), contradicting Theorem 2. Q.E.D.

References


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