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ON THE BOUNDEDNESS AND UNBOUNDEDNESS OF CERTAIN CONVOLUTION OPERATORS ON NILPOTENT LIE GROUPS

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ABSTRACT. One method of proving irreducibility of the "principal series" representations of semisimple Lie groups involves showing that a certain nonintegrable function on a nilpotent subgroup X cannot be regularized to give a bounded convolution operator on $L_2(X)$. This note gives an elementary proof of this unboundedness property for the groups X which occur in real-rank one semisimple groups.

Introduction. Let X be a connected, simply-connected nilpotent Lie group with a one-parameter group $\{\delta_r; 0<r<\infty\}$ of *dilations* and a *norm function* $|x|$, as defined by Knapp and Stein in [**2**]. For example, $X=\boldsymbol{R}^n$ with $\delta_r x=rx$ (scalar multiplication), and $|x|=\|x\|^n$, where $\|\cdot\|$ is the Euclidean norm. Our purpose in this note is to obtain some elementary results on the boundedness and unboundedness of regularizations of the nonintegrable function $|x|^{-1}$, acting by convolution on various function spaces. In particular, we obtain an independent proof of the "unboundedness theorem" of [**2**], i.e. the fact that no distribution obtained by regularization of $|x|^{-1}$ at the identity is a bounded convolution operator on $L_2(X)$. Our approach (suggested by the proof of the discrete version of this theorem in Hardy-Littlewood-Pólya [**1**, p. 214]) is to exploit the nonintegrability of $|x|^{-1}$ at infinity. This requires control over the singularity of $|x|^{-1}$ at the identity, which we gain by proving a simple "boundedness" theorem in §1. The general unboundedness theorem of Knapp and Stein is an easy corollary of the unboundedness of the "minimal" regularization of $|x|^{-1}$, which we establish in §2.

1. **Boundedness theorem.** Before stating the theorem, we recall some properties of the norm function $|x|$ (cf. [**2**, §§2 and 5]). We have $x\to|x|$ continuous on X and C^∞ on $X-\{e\}$ (we use e to denote the identity element of X), with $|x^{-1}|=|x|$ and $|x|>0$ if $x\neq e$. Under dilations it transforms by $|\delta_r x|=r^q|x|$, where $q>0$. If dx denotes a Haar measure on X, then

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$d(\delta_r x)=r^q\,dx$, so that $|x|^{-1}\,dx$ is invariant under $\{\delta_r\}$. The following inequalities hold:

$$(1)\qquad |xy| \leqq A \max\{|x|, |y|\},$$

$$(2)\qquad \big|\,|xy| - |y|\,\big| \leqq C_0\,|x|^d\,|y|^{1-d} \quad \text{if } |x| \leqq |y|.$$

Here $A>1$, while $0<d\leqq 1$. (If $\{p_i\}$ are the eigenvalues of the generator D of $\{\delta_r\}$, then $q=\sum p_i$ and $d=\min\{p_i\}/q$.) We also need the following integration formula: If $f(x)=\phi(|x|)$, where ϕ is a nonnegative Borel function on $\boldsymbol{R}^+=(0,\infty)$, then

$$(3)\qquad \int_X f(x)\,dx = \int_{\boldsymbol{R}^+} \phi(t)\,dt,$$

provided we normalize the Haar measure dx so that $\int_{1\leqq|x|\leqq e} |x|^{-1}\,dx=1$.

THEOREM 1. *Let σ be the distribution*

$$\sigma(f) = \int_{|x|\leqq 1} \{f(x) - f(e)\}\,|x|^{-1}\,dx,$$

$f \in C_0^\infty(X)$. Let $\mathscr{R}$ be the subspace of $C_0^\infty(X)$ consisting of all f of the form $f(x)=\phi(|x|)$, where $\phi \in C_0^\infty(\boldsymbol{R}^+)$. For such a function f, define

$$(4)\qquad N(f) = \left\{\int_0^\infty |\phi'(t)|^2\,(1+t^{2-d})\,dt\right\}^{1/2}$$

(d as in inequality (2)). Then there exists a constant $\beta>0$ so that

$$(5)\qquad \|\sigma * f\|_2 \leqq \beta N(f)$$

for all $f \in \mathscr{R}$.

PROOF. Define $(L_x f)(y)=f(x^{-1}y)$. Then by Schwarz' inequality,

$$(6)\qquad \|\sigma * f\|_2^2 \leqq C_\varepsilon \int_{|x|\leqq 1} \|L_x f - f\|_2^2 \frac{dx}{|x|^{1+\varepsilon}},$$

where $C_\varepsilon=\int_{|x|\leqq 1} |x|^{-1+\varepsilon}\,dx$, and $\varepsilon>0$. Suppose that $f \in \mathscr{R}$, $f(x)=\phi(|x|)$. Using the integration formula (3), to represent f as the integral of ϕ', and applying Schwarz' inequality again, we get the bound

$$(7)\qquad |f(x^{-1}y) - f(y)|^2 \leqq \big|\,|x^{-1}y| - |y|\,\big| \int_{[|x^{-1}y|,\,|y|]} |\phi'(t)|^2\,dt.$$

We will integrate (7) to obtain an estimate for the integrand in (6). Consider first the region $|y|\geqq A|x|$ (the constant A as in (1)). In this region $[|x^{-1}y|, |y|] \subseteq [A^{-1}|y|, A|y|]$ and (2) holds. Hence

$$\int_{|y|\geqq A|x|} |f(x^{-1}y) - f(y)|^2 \, dy \leqq C_0 \, |x|^d \int_{|y|\geqq A|x|} \left\{ \int_{A^{-1}|y|}^{A|y|} |\phi'(t)|^2 \, dt \right\} |y|^{1-d} \, dy$$
$$\leqq C_1 \, |x|^d \int_0^\infty |\phi'(t)|^2 \, t^{2-d} \, dt,$$

where we have interchanged the order of integration and used formula (3) again, and the constant C_1 is independent of ϕ. In the region $|y| \leqq A|x|$ we simply use the estimate $|x^{-1}y| \leqq A^2|x|$ to obtain from (7) the bound

$$\int_{|y|\leqq A|x|} |f(x^{-1}y) - f(y)|^2 \, dy \leqq 2A^2 \, |x| \int_0^\infty |\phi'(t)|^2 \, dt.$$

Assume now that $|x| \leqq 1$. Then $|x| \leqq |x|^d$, since $d \leqq 1$, so by the estimates just made,

$$\|L_x f - f\|_2^2 \leqq C_2 \, |x|^d \int_0^\infty |\phi'(t)|^2 \, (1 + t^{2-d}) \, dt, \tag{8}$$

C_2 a constant independent of ϕ. If we take $\varepsilon = d/2$ in (6) and use estimate (8), we obtain (5). Q.E.D.

2. **Unboundedness theorem.**

THEOREM 2. *Let T be the distribution*

$$T(f) = \sigma(f) + \int_{|x|\geqq 1} f(x) \, |x|^{-1} \, dx$$

(σ as in Theorem 1). Then left convolution by T is not bounded on $L_2(X)$.

PROOF. If ν is the distribution

$$\nu(f) = \int_{|x|\geqq 1} f(x) \, |x|^{-1} \, dx,$$

then it is easy to show that convolution by ν is not bounded on L_2. Indeed, let $\{\phi_n\} \subseteq C_0^\infty(\mathbf{R}^+)$ be a sequence of nonnegative functions such that $\phi_n(t) = t^{-1/2}(\log t)^{-1}$ in the interval $e \leqq t \leqq A^2 e^n$, with ϕ_n smoothly cut off outside this interval in such a way that $\sup_n \int_0^\infty \phi_n(t)^2 \, dt < \infty$ (this is certainly possible, since $\int_e^\infty t^{-1}(\log t)^{-2} \, dt < \infty$). Set $f_n(x) = \phi_n(|x|)$. By formula (3), $\{f_n\}$ is a bounded sequence in the L_2 norm. On the other hand, when $e \leqq |x| \leqq e^n$, then

$$(f_n * f_n)(x) \geqq \int_{A|x|\leqq|y|\leqq Ae^n} f_n(x^{-1}y) f_n(y) \, dy$$

But in the range $A|x|\leqq|y|$, $e\leqq|x|\leqq e^n$, one has also $e\leqq A^{-1}|y|\leqq|x^{-1}y|\leqq A|y|$. Hence from the definition of f_n we get the lower bound

$$(f_n * f_n)(x) \geqq C_1\int_{A|x|\leqq|y|\leqq Ae^n} |y|^{-1}(\log|y|)^{-2}\,dy$$
$$\geqq C_2(\log|x|)^{-1} \quad \text{if } |x| \leqq e^{n/2},$$

where $C_2>0$ is independent of n.

Integrating this estimate in the range $e\leqq|x|\leqq e^{n/2}$, we conclude that $\nu(f_n * f_n)\geqq C_2\log(n/2)$, and hence convolution by ν is not bounded on L_2. (*Remark.* $\nu\in L_p$ for all $p>1$, so by Young's inequality convolution by ν is bounded from L_r to L_2 for any $r<2$. The sequence $\{f_n\}$ was chosen to be bounded in L_2 but not bounded in L_r for any $r<2$.)

To prove Theorem 2, we only need to show that a sequence $\{f_n\}$ as above can be constructed with the additional property that $\sigma(f_n * f_n)$ remains bounded as $n\to\infty$. By Theorem 1 this will be the case provided that $\sup_n N(f_n)<\infty$. This can be easily done, however. The only delicate point is to truncate the functions ϕ_n slowly enough as $t\to\infty$ so that ϕ_n' is sufficiently small (this keeps $N(f_n)$ bounded), but yet rapidly enough to keep $\|f_n\|_2$ bounded. Specifically, we construct $\phi_n\in C_0^\infty(\boldsymbol{R}^+)$ as above, $\phi_n(t)=t^{-1/2}(\log t)^{-1}$ in $e\leqq t\leqq A^2e^n\equiv a_n$, requiring that $\phi_n(t)=0$ for $t>(1+e^{-nd})a_n\equiv b_n$, and taking a fixed cut-off in $1\leqq t\leqq e$. In the "cut-off" interval $[a_n, b_n]$ we require

$$0 \leqq \phi_n(t) \leqq Mn^{-1}e^{-n/2}, \qquad |\phi_n'(t)| \leqq M\exp[n(-3/2+d)],$$

where M is a fixed constant. This can certainly be done, since we can construct a C^1 function ψ_n with these properties (the estimate for the derivative being obtained by comparison with a linear cut-off in $[a_n, b_n]$), and then set $\phi_n=\theta * \psi_n$, where $\theta\in C_0^\infty(\boldsymbol{R})$ is fixed.

Suppose the sequence $\{\phi_n\}$ has been constructed, and set $f_n(x)=\phi_n(|x|)$. Then

$$\|f_n\|_2^2 \leqq C + \int_e^{a_n} t^{-1}(\log t)^{-2}\,dt + (b_n - a_n)M^2n^{-2}e^{-n}.$$

But $b_n-a_n=A^2e^{(1-d)n}$, and $d>0$, so $\sup_n\|f_n\|_2<\infty$. Furthermore,

$$N(f_n)^2 \leqq C + \int_e^{a_n} \frac{2(1+t^{2-d})}{t^3(\log t)^2}\,dt$$
$$+ (b_n - a_n)(1 + b_n^{2-d}) \sup_{a_n\leqq t\leqq b_n} |\phi_n'(t)|^2.$$

By construction of ϕ_n, the right side is bounded as $n\to\infty$, as required.

COROLLARY (KNAPP AND STEIN). *Let μ be any distribution on X such that $\mu=|x|^{-1}$ on $X\sim\{e\}$. Then convolution by μ is unbounded on $L_2(X)$.*

PROOF. Let T be the distribution of Theorem 2. Then $\mu-T$ vanishes on $X-\{e\}$, and hence is a finite linear combination of derivatives of the point mass at e. Pick a fixed, nonnegative $\phi=\phi^* \in C_0^\infty(X)$, and set $\phi_r(x)=(\log r)^{-1/2}r^{q/2}\phi(\delta_r x)$. Since $d(\delta_r x)=r^q\,dx$, while $|x|^{-1}\,dx$ is invariant under δ_r, we have

$$\begin{aligned}(\log r)T(\phi_r * \phi_r) &= \iint_{|x|\leq r^q} [\phi(xy) - \phi(y)]\phi(y)\,|x|^{-1}\,dx\,dy \\ &\quad + \iint_{|x|>r^q} \phi(y)\phi(xy)\,|x|^{-1}\,dx\,dy \\ &= T(\phi * \phi) - q(\log r)\,\|\phi\|_2^2.\end{aligned}$$

Also $\|\phi_r\|_2=(\log r)^{-1/2}\|\phi\|_2$. Hence $T(\phi_r * \phi_r)$ and $\|\phi_r\|_2$ are bounded as $r\to\infty$. On the other hand, if $\{X_i\}$ is an eigenvector for the differential of δ_e, then $(\partial/\partial X_i)\phi_r * \phi_r(e)=(\log r)^{-1}r^{p_i}(\partial\phi/\partial X_i) * \phi(e)$. Since all $p_i>0$, it is clear that $\mu(\phi_r * \phi_r)$ cannot be bounded as $r\to\infty$ unless $\mu-T$ is a multiple of the identity operator. Hence the assumption that convolution by μ is bounded on L_2 leads to the conclusion that convolution by T is bounded on L_2, contradicting Theorem 2. Q.E.D.

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