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CONVOLUTIONS AND ABSOLUTE CONTINUITY[1]

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ABSTRACT. We show that if E is a subset of the circle with positive Lebesgue measure, and g is integrable on almost every translate of E, then g is integrable on the whole circle. A generalization of this fact leads to a characterization of positive measures with nonvanishing absolutely continuous part.

Let T denote the multiplicative circle group with Haar measure m, and $tE=\{tx: x \in E\}$ be the translate of $E \subset T$ by $t \in T$. If $m(E)>0$ and $g \geqq 0$ is integrable on every translate of E, can one conclude that g is integrable? Equivalently, does $\chi_E * g < \infty$ everywhere imply $g \in L^1(T)$? If $\chi_E * g \in L^1(T)$, then Fubini's theorem instantaneously implies $g \in L^1(T)$. However, the remaining possibility requires more delicacy. An affirmative answer is a corollary of Theorem 1.

Everything here extends to arbitrary compact groups, and maybe even further.

We let P denote the class of nonzero, nonnegative, measurable functions on T, and M^+ the finite, positive regular Borel measures on T. The phrase "a.e." always refers to m.

LEMMA. *If $\nu \in M^+$, $m(E)>0$, and $0<\alpha<1$, then*

$$m\{t: \nu(tE) > \alpha m(E)\nu(T)\} \geqq (1-\alpha)[m(E)^{-1} - \alpha]^{-1}.$$

PROOF. Let $\varphi(t)=\nu(tE)$. Then $0 \leqq \varphi \leqq \nu(T)$ and $I=\int \varphi \, dm = m(E)\nu(T)$ by Fubini's theorem. Clearly $\varphi \leqq \alpha I + [\nu(T)-\alpha I]\chi_{\{\varphi>\alpha I\}}$, and integrating this gives the result.

THEOREM 1. *If μ is a σ-finite positive Borel measure, $f \in P$, and $f * \mu < \infty$ a.e., then $\mu(T)<\infty$.*

PROOF. We may assume $f=\chi_E$ for some set E of positive measure. Since μ is σ-finite, there are sets $F_n \uparrow T$ with $\mu(F_n)<\infty$. If we let $\mu_n=\mu|_{F_n}$, the Lemma shows that if $0<\alpha<1$ and $K_n=\{t: \mu_n(tE)>\alpha m(E)\mu_n(T)\}$, then

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$m(K_n) \geqq (1-\alpha)[m(E)^{-1}-\alpha]^{-1}$ for all n. Hence

$$m(\limsup K_n) \geqq (1-\alpha)[m(E)^{-1}-\alpha]^{-1} > 0,$$

and for almost all $t \in \limsup K_n$ we have $\infty > \mu(tE) = \lim \mu_n(tE) \geqq \limsup \alpha m(E)\mu_n(T)$. Thus $\mu(T) = \lim \mu_n(T) < \infty$.

COROLLARY 1. *If $g \in P$, $m(E)>0$, and $\chi_E * g < \infty$ a.e., then $g \in L^1(T)$.*

COROLLARY 2. *If $f, g \in P$, $f * g < \infty$ a.e., then $f, g \in L^1(T)$.*

PROOF OF THE COROLLARIES. When $g < \infty$ a.e., then $\mu = g\,dm$ is σ-finite and the corollaries follow from Theorem 1. If $m\{g=\infty\}>0$, an easy argument shows that both $\chi_E * g$ and $f * g$ are infinite on a set of positive measure.

REMARKS. 1. Some restriction on μ such as σ-finiteness is necessary in Theorem 1. For let $\mu(E)=0$ if E is of first category, and $\mu(E)=\infty$ otherwise. Then if E is of first category with $m(E)>0$, we have $\chi_E * \mu \equiv 0$ while $\mu(T)=\infty$.

2. A modification of the proof of Theorem 1 shows that if $m(E)>0$ and $g \in P$ is essentially bounded on almost every translate of E, then $g \in L^\infty(T)$.

Notice that Corollary 2 shows that if we assume $\mu \ll m$ in Theorem 1, then we can also conclude $f \in L^1(T)$. More generally, if $d\mu/dm = \mu_a \neq 0$, then $f * \mu_a \leqq f * \mu < \infty$ a.e. again guarantees $f \in L^1(T)$ by Corollary 2. We show this property characterizes measures with nonvanishing absolutely continuous part.

THEOREM 2. *Suppose $\mu \in M^+$. Then $\mu_a \neq 0$ if and only if whenever $f \in P$, $f * \mu < \infty$ a.e., we have $f \in L^1(T)$.*

PROOF. By the preceding paragraph we need only show that if $\mu_a = 0$, then there is an $f \in P \backslash L^1(T)$ with $f * \mu < \infty$ a.e. We will find $f_N \in P$ with $\int f_N\,dm = 1$, and $f_N * \mu < 2^{-N}$ except on a set of measure $< 2^{-N+1}$. Then $\sum_1^\infty f_N = f$ has the required properties.

Since μ is regular and supported on a null set S, there are disjoint closed sets $S_n \subset S$ with $\mu(T \backslash \bigcup_1^\infty S_n) = 0$. If $\mu_n = \mu|_{S_n}$, then $\sum_1^\infty \mu_n(T) = \mu(T) < \infty$. Choose M so that $\sum_{M+1}^\infty \mu_n(T) < 2^{-2N}$. Since the S_n are closed null sets, there is an interval I such that $H = \{xy : x \in I, y \in \bigcup_1^M S_n\}$ has measure $< 2^{-N}$. Let $f_N = m(I)^{-1}\chi_I$. Now $f_N * (\mu_1 + \cdots + \mu_M)$ is supported on H, so off H we have $f_N * \mu = f_N * (\mu_{M+1} + \cdots) = g$, say. But $\int g\,dm = \sum_{M+1}^\infty \mu_n(T) < 2^{-2N}$, so $m\{g > 2^{-N}\} < 2^{-N}$. Since clearly $\int f_N\,dm = 1$, the statements about f_N in the first paragraph are verified, completing the proof.

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