A CLASS OF UNIVALENT FUNCTIONS

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Abstract. A sharp coefficient estimate is obtained for a class $D(a)$ of functions univalent in the open unit disc. The radius of convexity and an arclength result are also determined for the class.

Let $D(a)$ denote the class of functions $f(z) = z + a_nz^n + \cdots$ analytic in the open unit disc $E$ and satisfying

$$|(f'(z) - 1)/(f'(z) + 1)| < a, \quad z \in E,$$

for some $a$, $0 < a \leq 1$. The values $f'(z)$ lie inside the circle in the right half plane with center $(1 + a)/(1 - a^2)$ and radius $2a/(1 - a^2)$. The class $D(a)$ is a subclass of the class of functions whose derivative has positive real part and hence a function in $D(a)$ is univalent in $E$. If $f \in D(a)$ it follows from Schwarz lemma that $f'(z) = (1 - az\theta(z))/(1 + az\theta(z))$, where $\theta(z)$ is analytic and $|\theta(z)| \leq 1$ in $E$.

A class of starlike functions has been studied by Padmanabhan [5] in which $f'(z)$ is replaced by $zf''(z)/f(z)$ in inequality (1).

A sharp coefficient estimate for the class $D(a)$ is proved in Theorem 1 using a technique of Clunie and Keogh [2]. In Theorem 2 the radius of convexity of the class is obtained and in Theorem 3 an arclength result is given.

Theorem 1. If $f(z) = z + \sum_{n=2}^{\infty} a_nz^n$ is in $D(a)$ for some $a$, $0 < a \leq 1$, then $|a_n| \leq 2a/n$, $n = 2, 3, \cdots$. The inequality is sharp.

Proof. Since $f(z)$ is in $D(a)$, then $f'(z) = (1 + az\theta(z))/(1 - az\theta(z))$, where $\theta(z) = \sum_{n=1}^{\infty} t_nz^n$ is analytic and $|\theta(z)| \leq 1$ for $z \in E$. Then

$$f'(z) - 1 = az\theta(z)(f'(z) + 1),$$

or

$$\sum_{n=2}^{\infty} n a_n z^{n-1} = \alpha \left( \sum_{n=0}^{\infty} t_n z^n \right) \left( 2z + \sum_{n=2}^{\infty} n a_n z^n \right).$$

Equating corresponding coefficients in (2) gives

$$na_n = \alpha \{(n - 1)t_0a_{n-1} + (n - 2)t_1a_{n-2} + \cdots + 2t_{n-3}a_2 + 2t_{n-2}\}.$$
Thus $a_n$ depends only on $a_2$, $a_3$, $\cdots$, $a_{n-1}$ and $\theta$ for $n \geq 2$. Hence, for $n \geq 2$, it follows from (2) that

$$\sum_{k=2}^{n} ka_k z^{k-1} + \sum_{k=n+1}^{\infty} b_k z^{k-1} = a \theta(z) \left\{ 2z + \sum_{k=2}^{n-1} ka_k z^k \right\},$$

which yields

$$\left| \sum_{k=2}^{n} ka_k z^{k-1} + \sum_{k=n+1}^{\infty} b_k z^{k-1} \right|^2 = a^2 |\theta(z)|^2 \left| 2z + \sum_{k=2}^{n-1} ka_k z^k \right|^2 \leq a^2 \left| 2z + \sum_{k=2}^{n-1} ka_k z^k \right|^2.$$

Integrating about $|z|=r$, $0<r<1$, gives

$$\sum_{k=2}^{\infty} k^2 |a_k|^2 r^{2k-2} + \sum_{k=n+1}^{\infty} |b_k|^2 r^{2k-2} \leq a^2 \left( 4r^2 + \sum_{k=2}^{n-1} k^2 |a_k|^2 r^{2k} \right).$$

If we take the limit as $r$ approaches 1, then

$$\sum_{k=2}^{n} k^2 |a_k|^2 \leq a^2 \left( 4 + \sum_{k=2}^{n-1} k^2 |a_k|^2 \right),$$

or

$$n^2 |a_n|^2 \leq 4a^2 + a^2 \sum_{k=2}^{n-1} k^2 |a_k|^2 - \sum_{k=2}^{n-1} k^2 |a_k|^2 = 4a^2 + (a^2 - 1) \sum_{k=2}^{n-1} k^2 |a_k|^2 \leq 4a^2,$$

since $a \leq 1$. Thus $|a_n| \leq 2a/n$ for $n \geq 2$.

Sharpness of the inequality is shown by

$$f(z) = \int_0^z \frac{1 + \alpha t^{n-1}}{1 - \alpha t^{n-1}} \alpha t = z + \frac{2\alpha z^n}{n} + \frac{2\alpha^2}{2n - 1} z^{2n-1} + \cdots.$$

**Theorem 2.** If $f(z)$ is in $D(\alpha)$, $0<\alpha \leq 1$, then

(i) $f(z)$ maps $|z|<(\sqrt{2} - 1)/\alpha$ onto a convex domain if

$$\left( \frac{\alpha^2 - 1}{\alpha^2} \right) \leq \alpha \leq 1,$$

(ii) $f(z)$ maps

$$|z| < \left( \frac{\alpha^2 - 1 + ((1 - \alpha^2)(1 + 4\alpha - \alpha^2))^{1/2}}{2\alpha(1 + \alpha)} \right)^{1/\alpha}$$

onto a convex domain if

$$0 < \alpha \leq \left( \frac{\alpha^2 - 1}{\alpha^2} \right)^{1/\alpha},$$

The bounds in (i) and (ii) are both sharp.
Proof. Since \( f(z) \) is in \( D(\alpha) \), we have
\[
f'(z) = \frac{1 + \alpha z \theta(z)}{1 - \alpha z \theta(z)},
\]
where \( \theta(z) \) is analytic and \( |\theta(z)| \leq 1 \) for \( z \in E \). Then
\[
\frac{f''(z)}{f'(z)} = \frac{2\alpha \{z \theta'(z) + \theta(z)\}}{1 - \alpha^2 z^2 |\theta(z)|^2}.
\]
But for functions \( \theta(z) \) [4, p. 168] we have
\[
|\theta'(z)| \leq (1 - |\theta(z)|^2)/(1 - |z|^2).
\]
Using this estimate we obtain
\[
\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{2\alpha |z| (|z| + |\theta(z)|)(1 - |z\theta(z)|)}{(1 - |z|^2)(1 - \alpha^2 |z|^2 |\theta(z)|^2)}.
\]
Therefore, \( |zf''(z)/f'(z)| \leq 1 \) provided
(3) \[ 2\alpha |z| (|z| + |\theta(z)|)(1 - |z\theta(z)|) \leq (1 - |z|^2)(1 - \alpha^2 |z|^2 |\theta(z)|^2). \]
Letting \( |z|=r \), \( |\theta(z)|=x \) and \( t=rx \), relation (3) becomes
(4) \[ 2\alpha r(r + tr^*) (1 - t) \leq (1 - r^2)(1 - \alpha^2 r^2). \]
We want to find the largest value of \( \rho \) such that (3) holds for all \( z \) such that \( |z|<\rho \) and for all \( \theta(z) \), \( |\theta(z)| \leq 1 \). This corresponds to finding the largest value of \( r \) for which (4) holds for all \( t, 0 \leq t \leq r \). Relation (4) becomes
(5) \[ H(t) = (\alpha^2 r^2 - \alpha^2 + 2\alpha t^2 + 2\alpha (r^2 - 1) t + (1 - 2\alpha r^2 - r^2) \geq 0. \]
We want to determine the largest value of \( r \) for which \( H(t) \geq 0 \), \( 0 \leq t \leq r \). Then \( f(z) \) will map \( |z|<r \) onto a convex domain. Since \( H'(t^*)=0 \) for \( t^*=(1-r^2)/[2-\alpha(1-r^2)] \) and \( H''(t^*)>0 \), \( H(t) \) assumes its minimum value at \( t^* \). We separate the proof into two cases:

Case A. \( r<t^* \). Now \( H(t) \) is nonincreasing on \([0, r]\), so \( H(t)>H(r) \) for \( 0 \leq t \leq r \). Since \( H(r)=(1-r^2)(-\alpha^2 r^2 - 2\alpha r + 1) \), \( H(r) \geq 0 \), provided \( \alpha^2 r^2 + 2\alpha r - 1 \leq 0 \) or \( r \leq (\sqrt{2}-1)/\alpha \). Thus, \( f(z) \) maps \( |z|<(\sqrt{2}-1)/\alpha \) onto a convex domain if \( r=(\sqrt{2}-1)/\alpha \leq t^* \). This restraint implies that \( \alpha \) must lie in the interval \([\alpha_0, 1]\), where \( \alpha_0=(\sqrt{2}-1)(\sqrt{3}+1)/\sqrt{2} \).

The function \( f(z)=-z-2/\alpha \log(1-\alpha z) \) shows this bound to be the best possible since
\[
1 + \frac{zf''(z)}{f'(z)} = \frac{1 + 2\alpha x z - \alpha^2 z^2}{1 - \alpha^2 z^2} = 0
\]
for \( z=(1-\sqrt{2})/\alpha \).
Case B. \( t^* \leq r \). The minimum value of \( H(t) \) on \([0, r]\) occurs at \( t^* \), so \( H(t) \geq H(t^*) \). Therefore (5) will be satisfied if \( H(t^*) \geq 0 \). This inequality reduces to

(6) \[ Q(r) \equiv \alpha(1 + \alpha)r^4 + (1 - \alpha^2)r^2 - (1 - \alpha) \leq 0. \]

But (6) is satisfied for \( r < r_1 \), where

\[ r_1 \equiv \frac{[\alpha^2 - 1 + ((1 - \alpha^2)(1 + 4\alpha - \alpha^2)^{1/2})/2\alpha(1 + \alpha)^{1/2}].} \]

We shall show that if \( 0 < \alpha < \alpha_0 \), then \( t^* \leq r_1 \). But \( t^* \leq r \), if and only if \( P(r) \equiv \alpha r^3 + r^2 + (2 - \alpha)r - 1 \geq 0 \). Denote the zero of \( P(r) \) in \((0, 1)\) by \( r_0 \). We shall show that \( r_1 > r_0 \) if \( 0 < \alpha < \alpha_0 \). A tedious calculation shows that if \( \alpha = \alpha_0 \),

\[ r_1 = \sqrt{2/(1 + \sqrt{3})} = r_0 = (\sqrt{2} - 1)/\alpha_0. \]

Also for a fixed \( r \) and \( \alpha < \alpha_0 \), the expression \( Q(r) \) increases with \( \alpha \). Thus, if (6) holds for a certain interval of values of \( r \) with \( \alpha = \alpha_1 \), then the condition holds for all \( \alpha < \alpha_1 \). Hence \( r_1 \) increases with decreasing \( \alpha \). But \( \alpha = \alpha_0 \) corresponds to \( r_1 = r_0 \). Thus for \( \alpha < \alpha_0, r_1 > r_0 \).

To show the estimate is sharp, we construct a function as follows. Let \( \beta \) be defined by

(7) \[ r_1(r_1 - \beta)/(1 - \beta r_1) = (1 - r_1^2)/[2 - \alpha(1 - r_1^2)]. \]

Since \( r_1 > r_0 \) for \( \alpha < \alpha_0 \), it follows that \( (1 - r_1^2)/[2 - \alpha(1 - r_1^2)] < r_1 \). Also, \( r_1 < 1 \). From (7) we have

\[ 0 < r_1(r_1 - \beta)/(1 - \beta r_1) < r_1, \]

or

\[ (r_1 - \beta)/(1 - \beta r_1) < 1. \]

This implies that \( (r_1^2 - 1)(1 - \beta^2) < 0 \). But \( r_1 < 1 \), so \( |\beta| < 1 \). Define \( \theta(z) \) by

(8) \[ \theta(z) = (z - \beta)/(1 - \beta z). \]

Since \( |\beta| < 1 \), \( |\theta(z)| \leq 1 \) for \( z \in E \). Define \( f(z) \) by

\[ f(z) = [1 - az\theta(z)]/[1 + az\theta(z)]. \]

Then

\[ 1 + zf''(z) = \frac{1 - \alpha^2z^2[\theta(z)]^2 - 2az\theta(z) - 2az^2\theta'(z)}{f'(z)} \]

where

\[ \theta'(z) = \{1 - [\theta(z)]^2\}/(1 - z^2). \]

Using (7) and (8) and remembering that \( Q(r_1) = 0 \), we have that \( 1 + r_1 f''(r_1)/f'(r_1) = 0 \). Therefore, \( f(z) \) is not convex in \( |z| < r \) if \( r > r_1 \).
Theorem 3. If \( f(z) \) is in \( D(\alpha) \) and if \( L_r(f) \) denotes the length of the image of \( |z|=r \) under \( f(z) \), \( 0<r<1 \), then \( L_r(f) = O\{\log(1/(1-\alpha r))\} \), as \( r \to 1 \).

Proof.

\[
L_r(f) = \left[ \int_0^{2\pi} |f'(re^{i\theta})| r \, d\theta \right]^{1/2}
\]

\[
\leq \int_0^{2\pi} \left| \frac{1 + \alpha re^{i\theta}}{1 - \alpha r e^{i\theta}} \right| r \, d\theta
\]

\[
= \int_0^{2\pi} \left| \frac{1 - \alpha \frac{r^2}{1 - 2ar \cos \theta + \alpha^2 r^2}}{1 - 2ar \cos \theta + \alpha^2 r^2} \right| r \, d\theta
\]

(9)

\[
= 2\pi r + 2r \int_0^{\pi} \frac{2ar \sin \theta}{1 - 2ar \cos \theta + \alpha^2 r^2} \, d\theta
\]

\[
= 2\pi r + 4r \log \frac{1 + \alpha r}{1 - \alpha r} = O\{\log 1/(1 - \alpha r)\}.
\]

The first integral in (9) is a Poisson integral and the second can be evaluated directly.

Remark. We state without proof that if \( f(z) \in D(\alpha) \) and if \( A_r(f) \) denotes the area of the image of \( |z|<r \) under \( f(z) \), \( 0<r<1 \), then

\[
A_r(f) \leq \pi \left\{ -3 - \frac{4}{\alpha^2 r^2} \log(1 - \alpha^2 r^2) \right\}.
\]

The inequality is sharp.

References


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