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A CLASS OF UNIVALENT FUNCTIONS

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ABSTRACT. A sharp coefficient estimate is obtained for a class $D(\alpha)$ of functions univalent in the open unit disc. The radius of convexity and an arclength result are also determined for the class.

Let $D(\alpha)$ denote the class of functions $f(z)=z+a_2z^2+\cdots$ analytic in the open unit disc E and satisfying

$$(1) \qquad |(f'(z)-1)/(f'(z)+1)|<\alpha, \qquad z\in E,$$

for some α, $0<\alpha\leqq 1$. The values $f'(z)$ lie inside the circle in the right half plane with center $(1+\alpha^2)/(1-\alpha^2)$ and radius $2\alpha/(1-\alpha^2)$. The class $D(\alpha)$ is a subclass of the class of functions whose derivative has positive real part and hence a function in $D(\alpha)$ is univalent in E. If $f\in D(\alpha)$ it follows from Schwarz lemma that $f'(z)=(1-\alpha z\theta(z))/(1+\alpha z\theta(z))$, where $\theta(z)$ is analytic and $|\theta(z)|\leqq 1$ in E.

A class of starlike functions has been studied by Padmanabhan **[5]** in which $f'(z)$ is replaced by $zf'(z)/f(z)$ in inequality (1).

A sharp coefficient estimate for the class $D(\alpha)$ is proved in Theorem 1 using a technique of Clunie and Keogh **[2]**. In Theorem 2 the radius of convexity of the class is obtained and in Theorem 3 an arclength result is given.

THEOREM 1. *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ is in $D(\alpha)$ for some α, $0<\alpha\leqq 1$, then $|a_n|\leqq 2\alpha/n$, $n=2,3,\cdots$. The inequality is sharp.*

PROOF. Since $f(z)$ is in $D(\alpha)$, then $f'(z)=(1+\alpha z\theta(z))/(1-\alpha z\theta(z))$, where $\theta(z)=\sum_{n=1}^{\infty}t_nz^n$ is analytic and $|\theta(z)|\leqq 1$ for $z\in E$. Then

$$f'(z)-1=\alpha z\theta(z)\{f'(z)+1\},$$

or

$$(2) \qquad \sum_{n=2}^{\infty}na_nz^{n-1}=\alpha\left(\sum_{n=0}^{\infty}t_nz^n\right)\left(2z+\sum_{n=2}^{\infty}na_nz^n\right).$$

Equating corresponding coefficients in (2) gives

$$na_n=\alpha\{(n-1)t_0a_{n-1}+(n-2)t_1a_{n-2}+\cdots+2t_{n-3}a_2+2t_{n-2}\}.$$

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Thus a_n depends only on $a_2, a_3, \cdots, a_{n-1}$ and θ for $n \geqq 2$. Hence, for $n \geqq 2$, it follows from (2) that

$$\sum_{k=2}^{n} k a_k z^{k-1} + \sum_{k=n+1}^{\infty} b_k z^{k-1} = \alpha\theta(z)\left\{2z + \sum_{k=2}^{n-1} k a_k z^k\right\},$$

which yields

$$\left|\sum_{k=2}^{n} k a_k z^{k-1} + \sum_{k=n+1}^{\infty} b_k z^{k-1}\right|^2 = \alpha^2 |\theta(z)|^2 \left|2z + \sum_{k=2}^{n-1} k a_k z^k\right|^2$$
$$\leqq \alpha^2 \left| 2z + \sum_{k=2}^{n-1} k a_k z^k\right|^2.$$

Integrating about $|z| = r$, $0 < r < 1$, gives

$$\sum_{k=2}^{n} k^2 |a_k|^2 r^{2k-2} + \sum_{k=n+1}^{\infty} |b_k|^2 r^{2k-2} \leqq \alpha^2\left\{4r^2 + \sum_{k=2}^{n-1} k^2 |a_k|^2 r^{2k}\right\}.$$

If we take the limit as r approaches 1, then

$$\sum_{k=2}^{n} k^2 |a_k|^2 \leqq \alpha^2\left\{4 + \sum_{k=2}^{n-1} k^2 |a_k|^2\right\},$$

or

$$n^2 |a_n|^2 \leqq 4\alpha^2 + \alpha^2 \sum_{k=2}^{n-1} k^2 |a_k|^2 - \sum_{k=2}^{n-1} k^2 |a_k|^2$$
$$= 4\alpha^2 + (\alpha^2 - 1) \sum_{k=2}^{n-1} k^2 |a_k|^2 \leqq 4\alpha^2,$$

since $\alpha \leqq 1$. Thus $|a_n| \leqq 2\alpha/n$ for $n \geqq 2$.

Sharpness of the inequality is shown by

$$f(z) = \int_0^z \frac{1 + \alpha t^{n-1}}{1 - \alpha t^{n-1}}\, \alpha t = z + \frac{2\alpha z^n}{n} + \frac{2\alpha^2}{2n-1} z^{2n-1} + \cdots.$$

THEOREM 2. *If $f(z)$ is in $D(\alpha)$, $0 < \alpha \leqq 1$, then*
(i) *$f(z)$ maps $|z| < (\sqrt{2} - 1)/\alpha$ onto a convex domain if*

$$\left(\frac{(\sqrt{2} - 1)(\sqrt{3} + 1)}{\sqrt{2}}\right) \leqq \alpha \leqq 1,$$

(ii) *$f(z)$ maps*

$$|z| < [\{\alpha^2 - 1 + ((1 - \alpha^2)(1 + 4\alpha - \alpha^2))^{1/2}\}/2\alpha(1 + \alpha)]^{1/2}$$

onto a convex domain if

$$0 < \alpha \leqq (\sqrt{2} - 1)(\sqrt{3} + 1)/\sqrt{2}.$$

The bounds in (i) *and* (ii) *are both sharp.*

PROOF. Since $f(z)$ is in $D(\alpha)$, we have

$$f'(z) = (1 + \alpha z\theta(z))/(1 - \alpha z\theta(z)),$$

where $\theta(z)$ is analytic and $|\theta(z)|\leqq 1$ for $z \in E$. Then

$$\frac{f''(z)}{f'(z)} = \frac{2\alpha\{z\theta'(z) + \theta(z)\}}{1 - \alpha^2 z^2[\theta(z)]^2}.$$

But for functions $\theta(z)$ [**4**, p. 168] we have

$$|\theta'(z)| \leqq (1 - |\theta(z)|^2)/(1 - |z|^2).$$

Using this estimate we obtain

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqq \frac{2\alpha|z|\,(|z| + |\theta(z)|)(1 - |z\theta(z)|)}{(1 - |z|^2)(1 - \alpha^2\,|z|^2\,|\theta(z)|^2)}.$$

Therefore, $|zf''(z)/f'(z)|\leqq 1$ provided

(3) $$2\alpha\,|z|\,(|z| + |\theta(z)|)(1 - |z\theta(z)|) \leqq (1 - |z|^2)(1 - \alpha^2\,|z|^2\,|\theta(z)|^2).$$

Letting $|z|=r$, $|\theta(z)|=x$ and $t=rx$, relation (3) becomes

(4) $$2\alpha r(r + tr^{-1})(1 - t) \leqq (1 - r^2)(1 - \alpha^2 t^2).$$

We want to find the largest value of ρ such that (3) holds for all z such that $|z|<\rho$ and for all $\theta(z)$, $|\theta(z)|\leqq 1$. This corresponds to finding the largest value of r for which (4) holds for all t, $0\leqq t\leqq r$. Relation (4) becomes

(5) $$H(t) \equiv (\alpha^2 r^2 - \alpha^2 + 2\alpha)t^2 + 2\alpha(r^2 - 1)t + (1 - 2\alpha r^2 - r^2) \geqq 0.$$

We want to determine the largest value of r for which $H(t)\geqq 0$, $0\leqq t\leqq r$. Then $f(z)$ will map $|z|<r$ onto a convex domain. Since $H'(t^*)=0$ for $t^*=(1-r^2)/[2-\alpha(1-r^2)]$ and $H''(t)>0$, $H(t)$ assumes its minimum value at t^*. We separate the proof into two cases:

Case A. $r<t^*$. Now $H(t)$ is nonincreasing on $[0, r]$, so $H(t)>H(r)$ for $0\leqq t\leqq r$. Since $H(r)=(1-r^2)(-\alpha^2r^2-2\alpha r+1)$, $H(r)\geqq 0$, provided $\alpha^2r^2+2\alpha r-1\leqq 0$ or $r\leqq(\sqrt{2}-1)/\alpha$. Thus, $f(z)$ maps $|z|<(\sqrt{2}-1)/\alpha$ onto a convex domain if $r=(\sqrt{2}-1)/\alpha\leqq t^*$. This restraint implies that α must lie in the interval $[\alpha_0, 1]$, where $\alpha_0\equiv(\sqrt{2}-1)(\sqrt{3}+1)/\sqrt{2}$.

The function $f(z)=-z-2/\alpha\log(1-\alpha z)$ shows this bound to be the best possible since

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 + 2\alpha z - \alpha^2 z^2}{1 - \alpha^2 z^2} = 0$$

for $z=(1-\sqrt{2})/\alpha$.

Case B. $t^*\leqq r$. The minimum value of $H(t)$ on $[0, r]$ occurs at t^*, so $H(t)\geqq H(t^*)$. Therefore (5) will be satisfied if $H(t^*)\geqq 0$. This inequality reduces to

$$Q(r) \equiv \alpha(1+\alpha)r^4 + (1-\alpha^2)r^2 - (1-\alpha) \leqq 0. \tag{6}$$

But (6) is satisfied for $r<r_1$, where

$$r_1 \equiv [\{\alpha^2 - 1 + ((1-\alpha^2)(1+4\alpha-\alpha^2))^{1/2}\}/2\alpha(1+\alpha)]^{1/2}.$$

We shall show that if $0<\alpha<\alpha_0$, then $t^*\leqq r_1$. But $t^*\leqq r$, if and only if $P(r)\equiv\alpha r^3+r^2+(2-\alpha)r-1\geqq 0$. Denote the zero of $P(r)$ in $(0, 1)$ by r_0. We shall show that $r_1>r_0$ if $0<\alpha<\alpha_0$. A tedious calculation shows that if $\alpha=\alpha_0$,

$$r_1 = \sqrt{2}/(1+\sqrt{3}) = r_0 = (\sqrt{2}-1)/\alpha_0.$$

Also for a fixed r and $\alpha<\alpha_0$, the expression $Q(r)$ increases with α. Thus, if (6) holds for a certain interval of values of r with $\alpha=\alpha_1$, then the condition holds for all $\alpha<\alpha_1$. Hence r_1 increases with decreasing α. But $\alpha=\alpha_0$ corresponds to $r_1=r_0$. Thus for $\alpha<\alpha_0$, $r_1>r_0$.

To show the estimate is sharp, we construct a function as follows. Let β be defined by

$$r_1(r_1-\beta)/(1-\beta r_1) = (1-r_1^2)/[2-\alpha(1-r_1^2)]. \tag{7}$$

Since $r_1>r_0$ for $\alpha<\alpha_0$, it follows that $(1-r_1^2)/[2-\alpha(1-r_1^2)]<r_1$. Also, $r_1<1$. From (7) we have

$$0 < r_1(r_1-\beta)/(1-\beta r_1) < r_1,$$

or

$$(r_1-\beta)/(1-\beta r_1) < 1.$$

This implies that $(r_1^2-1)(1-\beta^2)<0$. But $r_1<1$, so $|\beta|<1$. Define $\theta(z)$ by

$$\theta(z) = (z-\beta)/(1-\beta z). \tag{8}$$

Since $|\beta|<1$, $|\theta(z)|\leqq 1$ for $z\in E$. Define $f(z)$ by

$$f(z) = [1-\alpha z\theta(z)]/[1+\alpha z\theta(z)].$$

Then

$$1+\frac{zf''(z)}{f'(z)} = \frac{1-\alpha^2z^2[\theta(z)]^2 - 2\alpha z\theta(z) - 2\alpha z^2\theta'(z)}{1-\alpha^2z^2[\theta(z)]^2},$$

where

$$\theta'(z) = \{1-[\theta(z)]^2\}/\{1-z^2\}.$$

Using (7) and (8) and remembering that $Q(r_1)=0$, we have that $1+r_1f''(r_1)/f'(r_1)=0$. Therefore, $f(z)$ is not convex in $|z|<r$ if $r>r_1$.

THEOREM 3. *If $f(z)$ is in $D(\alpha)$ and if $L_r(f)$ denotes the length of the image of $|z|=r$ under $f(z)$, $0<r<1$, then $L_r(f)=O\{\log(1/(1-\alpha r))\}$, as $r\to 1$.*

PROOF.

$$\begin{aligned}
L_r(f) &= \int_{|z|=r} |f'(z)|\,|dz| = \int_0^{2\pi} |f'(re^{i\theta})|\, r\, d\theta \\
&\leqq \int_0^{2\pi} \left| \frac{1+\alpha r e^{i\theta}}{1-\alpha r^{i\theta}} \right| r\, d\theta \\
&= \int_0^{2\pi} \left| \frac{1-\alpha^2 r^2 + 2\alpha r\, i \sin\theta}{1-2\alpha r\cos\theta + \alpha^2 r^2} \right| r\, d\theta \\
&\leqq r \int_0^{2\pi} \frac{1-(\alpha r)^2}{1-2\alpha r\cos\theta+(\alpha r)^2}\, d\theta + \int_0^{2\pi} \frac{2\alpha r^2 |\sin\theta|}{1-2\alpha r\cos\theta+\alpha^2 r^2}\, d\theta \qquad (9) \\
&= 2\pi r + 2r \int_0^{\pi} \frac{2\alpha r \sin\theta}{1-2\alpha r\cos\theta+\alpha^2 r^2}\, d\theta \\
&= 2\pi r + 4r \log \frac{1+\alpha r}{1-\alpha r} = O\{\log 1/(1-\alpha r)\}.
\end{aligned}$$

The first integral in (9) is a Poisson integral and the second can be evaluated directly.

REMARK. We state without proof that if $f(z) \in D(\alpha)$ and if $A_r(f)$ denotes the area of the image of $|z|<r$ under $f(z)$, $0<r<1$, then

$$A_r(f) \leqq \pi\{-3 - (4/\alpha^2 r^2)\log(1-\alpha^2 r^2)\}.$$

The inequality is sharp.

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