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S-ALGEBRAS ON SETS IN C^n

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ABSTRACT. We give conditions which are necessary and sufficient for polynomial approximation of any continuous function on a compact subset of C^n.

Let X be a compact set in C^n, complex n-space, $P(X)$ the uniform closure of the polynomials on X, $C(X)$ all continuous functions on X, m_{2n} $2n$-dimensional Lebesgue measure on C^n, and for any λ in C^n let $E(\lambda)=\{z \in C^n | z_i=\lambda_i \text{ for some } i\}$.

A given set is a *strong peak set* if it is an intersection of peak sets and meets the boundary of each of them in a set which contains no nonempty perfect subsets. We say a Banach algebra A is an *S-algebra* if when x is in A and $\hat{x}$, the Gelfand transform of x, vanishes at some p, then there exist x_n in A such that $\hat{x}_n$ vanish in (perhaps different) neighborhoods of p and $\|x_n-x\|\to 0$. For example, for any locally compact abelian group G, $L^1(G)$ is an S-algebra [**6**, p. 51]. The main question which motivates us here is: If A is a uniform algebra on a compact space X and A is an S-algebra, does $A=C(X)$? Our main result is the following.

THEOREM. *A necessary and sufficient condition that $P(X)=C(X)$ is that* (i) *$P(X)$ is an S-algebra*, (ii) *for almost all $\lambda \in C^n$ with respect to m_{2n}, $E(\lambda)\cap X$ is a strong peak set, and* (iii) *each point of X is a peak point for $P(X)$.*

We begin with some observations about uniform algebras which are S-algebras.

LEMMA 1. *Let A be a uniform algebra on a compact space X and suppose that A is an S-algebra. Then*: (i) *The maximal ideal space of A is X.* (ii) *A is normal.* (iii) *If each point of X is a peak point then A satisfies condition D* [**4**, p. 86], *i.e. if $f \in A$ and $f(p)=0$ then there exist $f_n \in A$ vanishing on neighborhoods of p such that $f_n f\to f$.*

PROOF. (i) Let p be a homomorphism on A and μ_p a representing measure for p with minimal closed support. If μ_p is not a point-mass then

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some $q \neq p$ lies in its closed support. Find f in A such that $f(p)=1$ and $f(q)=0$. Since A is an S-algebra we can assume that f vanishes in a neighborhood of q. Thus $f\mu_p$ is a complex representing measure for p, and since it dominates a (positive) representing measure for p [3, p. 33], we have a contradiction to the minimality of μ_p.

(ii) By part (i), to show normality of A we need only show regularity. But if $p \neq q$ then as above there is an f in A such that f vanishes on a neighborhood of p and $f(q)=1$. If K is compact and $q \notin K$ then by compactness one finds a function f in A such that $f=0$ on K and $f(q)=1$.

(iii) Suppose k peaks at p. Then there exist g_n in A such that g_n vanish on neighborhoods of p such that $\|g_n-(1-k^n)\| \to 0$. Hence, $\|f-fg_n\| \leqq \|f(1-k^n-g_n)\|+\|fk^n\| \to 0$ so that $fg_n \to f$.

Part (iii) allows us to do spectral synthesis on the maximal ideal space of any uniform S-algebra as follows.

LEMMA 2. *Let A be a uniform algebra which is an S-algebra on X and let I be a closed ideal of A. If each point of X is a peak point for A then I contains every element f in A such that $\partial\{x \mid f(x)=0\} \cap \operatorname{hull}(I)$ contains no nonempty perfect set.*

PROOF. Since A is normal and satisfies condition D, this is immediate from [4, p. 86].

We shall also need the following lemma which generalizes a result in [7] from one variable. A detailed proof is given in [1].

LEMMA 3. *Let X be a compact set in C^n and let μ be a regular bounded Borel measure on X. Let*

$$\hat{\mu}(z) = \int \frac{d\mu(\lambda)}{(\lambda_1 - z_1) \cdots (\lambda_n - z_n)}$$

and

$$N_\mu(z) = \int \frac{d\,|\mu|\,(\lambda)}{|\lambda_1 - z_1| \cdots |\lambda_n - z_n|}\ .$$

Then $N_\mu(z) < \infty$ a.e. with respect to m_{2n} and if $\hat{\mu}(z)=0$ a.e. m_{2n} then $\mu=0$.

PROOF OF THE THEOREM. Let $E_1(X)=\bigcup\{E(z) \mid z \in X\}$. Let μ be a measure on X such that $\mu \perp P(X)$. We must show that $\mu=0$. Now clearly if $z \notin E_1(X)$ then $\hat{\mu}(z)=0$. Now call $E(X)$ the set of z for which $E(z) \cap X$ is a strong peak set and for which $N_\mu(z)<\infty$. Since this only differs from $E_1(X)$ by a set of measure 0, we need only show that $\hat{\mu}$ vanishes on $E(X)$. Now if $\lambda \in E(X)$, we know that $E(\lambda) \cap X = \bigcap_{i=1}^\infty K_i$ with k_i peaking on K_i and $E(\lambda) \cap \partial K_i$ contains no nonempty perfect subset. Note that the hull of the closed ideal generated by $(z_1-\lambda_1) \cdots (z_n-\lambda_n)$ is $E(\lambda) \cap X$ so that, by Lemma 2, $1-k_i^{n_i} \in$ the uniform closure of $P(X)(z_1-\lambda_1) \cdots (z_n-\lambda_n)$

for any positive n_i. Now choose n_i so that $k_i^{n_i} \to \chi_{E(\lambda)}$ boundedly pointwise on X. Then find g_j in $P(X)$ such that $\|g_j(z_1-\lambda_1)\cdots(z_n-\lambda_n)+1-k_j^{n_j}\| \to 0$. In other words, $f_j = 1+g_j(z_1-\lambda_1)\cdots(z_n-\lambda_n) \to \chi_{E(\lambda)}$ boundedly pointwise on X. Since $N_\mu(\lambda)<\infty$, $|\mu|$ vanishes on $E(\lambda)$. Also as $j\to\infty$,

$$\frac{f_j}{(\lambda_1 - z_1)\cdots(\lambda_n - z_n)} \to 0$$

pointwise on $X-E(\lambda)$, and dominatedly. Hence

$$\hat{\mu}(\lambda) = \int \frac{f_j}{(\lambda_1 - z_1)\cdots(\lambda_n - z_n)}\, d\mu \to 0 \quad \text{as } j \to \infty,$$

so $\hat{\mu}(\lambda)=0$. Thus $\hat{\mu}=0$ a.e. and, by Lemma 3, $\mu=0$ and the theorem is proved.

For a uniform algebra A and a point x in $M(A)$, the maximal ideal space of A, call the *0-germ at x* the set of functions in A which vanish on a neighborhood of x. We close with an example of a uniform algebra A such that for each point x in a dense set in $M(A)$ the 0-germ is dense in the maximal ideal determined by x. In other words the S-algebra condition is satisfied on at least a dense subset. McKissick [5] has proved the following.

LEMMA 4. *Let D be the open unit disk. Then there is a sequence* $\{a_k\}$ *in D*, $0<|a_k|\leqq|a_{k+1}|\to 1$, *such that for any* $\varepsilon'>0$ *there is a sequence* $\{J_k\}$ *of open disks in D centered at* $\{a_k\}$ *respectively such that*:

(1) $\sum_1^\infty$ *length*$(\partial J_k)<\varepsilon'$.

(2) *There exist rational functions* r_n *with poles at* $a_1,\cdots,a_n$ *such that* $r_n\to f$ *uniformly on* $(\bigcup_{k=1}^\infty J_k)'$ *and* $f=0$ *on* D' *while* $f(0)=1$.

Using the above lemma we prove the following.

LEMMA 5. *Let* $c=|a_1|/2$. *There is a constant* $M>0$ *such that for any positive* ε, δ *there is a* δ' *and* $\{D_k\}$ *a sequence of open disks in* $N(0,\delta'/c)-N(0,\delta'c)$ *such that*:

(1) $\sum_1^\infty$ *length*$(\partial D_k)<\delta'c$.

(2) *There exist rational functions* $\{r_n\}$ *with poles in* $D_1\cup\cdots\cup D_n$ *such that* $r_n\to g$ *uniformly on* $(\bigcup_{k=1}^\infty D_k)'$ *and*

(i) $|g|\leqq M$ *on* $(\bigcup_1^\infty D_k)'$,

(ii) $g=0$ *on* $N(0,\delta')$,

(iii) $|1-g|<\varepsilon$ *on* $N(0,\delta)'$.

In fact if f is the function obtained by Lemma 1 with ε' a fixed constant (to be determined) independent of ε and δ, then δ' can be chosen as $\delta\delta(\varepsilon)$ where $\delta(\varepsilon)$ is a function such that $|z|<\delta(\varepsilon)$ implies $|1-f(z)|<\varepsilon$.

PROOF. For disks $\{J_k\}$ which we now choose in D let $\{D_k\}$ be their respective images under the map $1/cz$. Since $|a_k|\geqq 2c$, by taking a sufficiently small ε' we can choose the open disks J_k guaranteed by Lemma 1 so that $z \in \bigcup J_k$ implies $|z| > c$ and so that $\sum \text{length}(\partial D_k) < 1$. Thus $D_k \subset N(0, 1/c^2) - N(0, 1)$ for all k. Let f denote the limit on $(\bigcup J_k)'$ of the rational functions guaranteed by Lemma 1, and let M be the maximum of f on this set. Now since $f(0)=1$, $|z| < \delta(\varepsilon)$ implies $|1-f(z)| < \varepsilon$. Set $\delta' = \delta\delta(\varepsilon)$ and let $g(z) = f(\delta'/zc)$. Then redefining D_k as $\delta' D_k$ we have $D_k \subset N(0, \delta'/c^2) - N(0, \delta')$, $g(z)$ is obviously defined for $z \notin D_k$, and

(1) $\sum \text{length}(\partial D_k) < \delta'$,

(2) (i) $|g| < M$ on $(\bigcup D_k)'$,

(ii) $g(z) = 0$ on $N(0, \delta'/c)$ since $|\delta'/zc| > 1$ there, and

(iii) $|1-g(z)| < \varepsilon$ on $N(0, \delta/c)'$ since $|\delta'/zc| < \delta(\varepsilon)$ there.

The statement of the lemma follows by replacing δ in the above by δc.

COROLLARY. *There is a constant M such that given positive δ', ε there exist D_k and g as in the above lemma satisfying* (1) *and* (2) *if δ is taken as $\delta'/\delta(\varepsilon)$.*

Of course the above lemmas hold with 0 replaced by any point p. Also since the function $f(z) = \sum_1^\infty 1/[\phi'(a_k)(z-a_k)]$ used by McKissick in Lemma 1 has a $\delta(\varepsilon) < \beta\varepsilon$ for some fixed β and small enough ε we see that $\delta(\varepsilon)$ in the above statements can be replaced by ε. We now construct the example. Pick $m > 1$ such that $2^m c > 1$. Let $X_{m-1} = D$ and $S_{m-1} = \phi$. Define $S_n \subset X_n$, $\{D_k^{j,n}\}$, for $n \geqq m$ inductively as follows. Suppose that $S_{n-1} = \{a_1, \cdots, a_k\}$. Choose other points $a_{k+1}, \cdots, a_t$ in X_{n-1} so that each point of X_{n-1} is within $1/2^n$ of some a_i, and let $S_n = \{a_1, \cdots, a_t\}$. Let d denote the minimum distance between the points of S_n. Letting $\delta = \varepsilon = d/(2^{n+j}c^{1/2})$ find $\{D_k^{j,n}\}_{k=1}^\infty$ open disks in $N(a_j, \delta\varepsilon/c) - N(a_j, \delta\varepsilon c)$ such that $\sum_{k=1}^\infty \text{length}(\partial D_k^{j,n}) < d^2/4^{n+j} < 1/2^{n+j}$ and (2) holds. Let $X_n = X_{n-1} - \bigcup_{k,j} D_k^{j,n}$. Observe that since $\delta\varepsilon/c < d$ we have $S_n \subset X_n$. Note too that $\sum_{k,j=1}^\infty \text{length}(\partial D_k^{j,n}) < 1/2^n$ so that if we set $X = \bigcap_{n=m}^\infty X_n$, we have excised a countable number of discs whose boundaries have total length <1. Thus by Lemma 1 of [5], $R(X) \subsetneq C(X)$. It is now clear that given any $\varepsilon > 0$ and any a_j some $N(a_j, d/(2^{n+j}c^{1/2})) \subset N(a_j, \varepsilon)$ so there is a g in $R(X)$ so that $\|g\| \leqq M$, g vanishes on a neighborhood of a_j and $|1-g| < \varepsilon$ on $N(a_j, \varepsilon)'$. Thus the 0-germ at a_j is pointwise boundedly dense in the maximal ideal at a_j and so is dense. Since the $\{a_j\}$ are a dense subset of X the example has the required properties.

Can the example be altered so that it is an S-algebra? One's first inclination is to cover the disk by smaller and smaller δ_n' neighborhoods given by the Corollary, but clearly it is not possible to do this and even retain $\sum \delta_n' < \infty$. However the example is rather simple-minded in that the same function is used over and over. Perhaps a choice of other functions will extend the example. Some questions raised by the above are: (1) If the 0-germ at p is dense in the maximal ideal determined by p, is p a peak point? (2) Is the example normal? (3) From an example of Cole (see also Basener [2]), it is well known that (iii) alone is not sufficient to imply the conclusion of the theorem. Are any of the hypotheses of the theorem redundant?

Wilken [8] has shown that if a uniform algebra A is an S-algebra on $[0, 1]$ then $A=C[0, 1]$. In closing we also show the following.

THEOREM. *If A is a uniform algebra and A is an S-algebra on the unit circle T, then $A=C(T)$.*

PROOF. Let p, q be peak points for A in T, so $\{p, q\}$ is a peak set. Let f in A peak there. Then there are g_n vanishing on neighborhoods of p and h_n vanishing on neighborhoods of q such that $\|(1-f^n)-g_n\|<1/n$ and $\|(1-f^n)-h_n\|<1/n$ with h_n and g_n in A. Then $\|(1-f_n)^2-h_n g_n\|<5/n$. Let $k_n=0$ on one of the arcs $[p, q]$ joining p to q and let $k_n=h_n g_n$ on the other arc $[q, p]$. Then because A is normal and hence local, k_n are in A. But $k_n \to \chi_{(q,p)}$ boundedly pointwise. Thus if $\mu \in A^\perp$, $\mu_{(q,p)}=\mu_{[q,p]} \in A^\perp$. Hence $[q, p]$ is a peak set. Since every closed interval is an intersection of such peak sets, it follows that every closed set is a peak set and thus $A=C(T)$.

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