S-ALGEBRAS ON SETS IN C^n
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ABSTRACT. We give conditions which are necessary and sufficient for polynomial approximation of any continuous function on a compact subset of C^n.

Let X be a compact set in C^n, complex n-space, P(X) the uniform closure of the polynomials on X, C(X) all continuous functions on X, \( m_{2n} \) 2n-dimensional Lebesgue measure on C^n, and for any \( \lambda \) in C^n let \( E(\lambda) = \{ z \in C^n : z_i = \lambda_i \text{ for some } i \} \).

A given set is a strong peak set if it is an intersection of peak sets and meets the boundary of each of them in a set which contains no nonempty perfect subsets. We say a Banach algebra A is an S-algebra if when x is in A and \( \hat{x} \), the Gelfand transform of x, vanishes at some p, then there exist \( x_n \) in A such that \( x_n \) vanish in (perhaps different) neighborhoods of p and \( \| x_n - x \| \to 0 \). For example, for any locally compact abelian group G, \( L^1(G) \) is an S-algebra [6, p. 51]. The main question which motivates us here is: If A is a uniform algebra on a compact space X and A is an S-algebra, does A = C(X)? Our main result is the following.

**Theorem.** A necessary and sufficient condition that P(X) = C(X) is that (i) P(X) is an S-algebra, (ii) for almost all \( \lambda \in C^n \) with respect to \( m_{2n} \), \( E(\lambda) \cap X \) is a strong peak set, and (iii) each point of X is a peak point for P(X).

We begin with some observations about uniform algebras which are S-algebras.

**Lemma 1.** Let A be a uniform algebra on a compact space X and suppose that A is an S-algebra. Then: (i) The maximal ideal space of A is X. (ii) A is normal. (iii) If each point of X is a peak point then A satisfies condition D [4, p. 86], i.e. if \( f \in A \) and \( f(p) = 0 \) then there exist \( f_n \in A \) vanishing on neighborhoods of p such that \( f_n f \to f \).

**Proof.** (i) Let \( p \) be a homomorphism on A and \( \mu_p \) a representing measure for \( p \) with minimal closed support. If \( \mu_p \) is not a point-mass then

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some \( q \neq p \) lies in its closed support. Find \( f \) in \( A \) such that \( f(p) = 1 \) and \( f(q) = 0 \). Since \( A \) is an \( S \)-algebra we can assume that \( f \) vanishes in a neighborhood of \( q \). Thus \( f\mu_x \) is a complex representing measure for \( p \), and since it dominates a (positive) representing measure for \( p \) [3, p. 33], we have a contradiction to the minimality of \( \mu_x \).

(ii) By part (i), to show normality of \( A \) we need only show regularity. But if \( p \neq q \) then as above there is an \( f \) in \( A \) such that \( f \) vanishes on a neighborhood of \( p \) and \( f(q) = 1 \). If \( K \) is compact and \( q \notin K \) then by compactness one finds a function \( f \) in \( A \) such that \( f = 0 \) on \( K \) and \( f(q) = 1 \).

(iii) Suppose \( k \) peaks at \( p \). Then there exist \( g_n \) in \( A \) such that \( g_n \) vanish on neighborhoods of \( p \) such that \( \| g_n - (1 - k^n) \| \to 0 \). Hence, \( \| f - fg_n \| \leq \| f(1 - k^n - g_n) \| + \| fk^n \| \to 0 \) so that \( fg_n \to f \).

Part (iii) allows us to do spectral synthesis on the maximal ideal space of any uniform \( S \)-algebra as follows.

**Lemma 2.** Let \( A \) be a uniform algebra which is an \( S \)-algebra on \( X \) and let \( I \) be a closed ideal of \( A \). If each point of \( X \) is a peak point for \( A \) then \( I \) contains every element \( f \) in \( A \) such that \( \partial \{ x \mid f(x) = 0 \} \cap \text{hull}(I) \) contains no nonempty perfect set.

**Proof.** Since \( A \) is normal and satisfies condition D, this is immediate from [4, p. 86].

We shall also need the following lemma which generalizes a result in [7] from one variable. A detailed proof is given in [1].

**Lemma 3.** Let \( X \) be a compact set in \( C^n \) and let \( \mu \) be a regular bounded Borel measure on \( X \). Let

\[
\hat{\mu}(z) = \int \frac{d\mu(\lambda)}{(\lambda_1 - z_1) \cdots (\lambda_n - z_n)}
\]

and

\[
N_\mu(z) = \int \frac{d|\mu|(\lambda)}{|\lambda_1 - z_1| \cdots |\lambda_n - z_n|}.
\]

Then \( N_\mu(z) < \infty \) a.e. with respect to \( m_2^n \) and if \( \hat{\mu}(z) = 0 \) a.e. \( m_2^n \) then \( \mu = 0 \).

**Proof of the Theorem.** Let \( E_1(X) = \bigcup \{ E(z) \mid z \in X \} \). Let \( \mu \) be a measure on \( X \) such that \( \mu \perp P(X) \). We must show that \( \mu = 0 \). Now clearly if \( z \notin E_1(X) \) then \( \hat{\mu}(z) = 0 \). Now call \( E(X) \) the set of \( z \) for which \( E(z) \cap X \) is a strong peak set and for which \( N_\mu(z) < \infty \). Since this only differs from \( E_1(X) \) by a set of measure 0, we need only show that \( \hat{\mu} \) vanishes on \( E(X) \). Now if \( \lambda \in E(X) \), we know that \( E(\lambda) \cap X = \bigcap_{i=1}^c K_i \) with \( K_i \) peaking on \( K_i \) and \( E(\lambda) \cap \partial K_i \) contains no nonempty perfect subset. Note that the hull of the closed ideal generated by \((z_1 - \lambda_1) \cdots (z_n - \lambda_n)\) is \( E(\lambda) \cap X \) so that, by Lemma 2, \( 1 - k_i^n \) is in the uniform closure of \( P(X)(z_1 - \lambda_1) \cdots (z_n - \lambda_n) \)
for any positive \( n_i \). Now choose \( n_i \) so that \( k_i^{n_i} \rightarrow \chi_{E(A)} \) boundedly pointwise on \( X \). Then find \( g_j \) in \( P(X) \) such that \( \| g_j(z_1 - \lambda_1) \cdots (z_n - \lambda_n) + 1 - k_j \| \rightarrow 0 \). In other words, \( f_j = 1 + g_j(z_1 - \lambda_1) \cdots (z_n - \lambda_n) \rightarrow \chi_{E(A)} \) boundedly pointwise on \( X \). Since \( N_d(\lambda) < \infty \), \( |\mu| \) vanishes on \( E(\lambda) \). Also as \( j \rightarrow \infty \),

\[
\frac{f_j}{(\lambda_1 - z_1) \cdots (\lambda_n - z_n)} \rightarrow 0
\]

pointwise on \( X - E(\lambda) \), and dominatedly. Hence

\[
\hat{\mu}(\lambda) = \int \frac{f_j}{(\lambda_1 - z_1) \cdots (\lambda_n - z_n)} d\mu \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty,
\]

so \( \hat{\mu}(\lambda) = 0 \). Thus \( \hat{\mu} = 0 \) a.e. and, by Lemma 3, \( \mu = 0 \) and the theorem is proved.

For a uniform algebra \( A \) and a point \( x \) in \( M(A) \), the maximal ideal space of \( A \), call the 0-germ at \( x \) the set of functions in \( A \) which vanish on a neighborhood of \( x \). We close with an example of a uniform algebra \( A \) such that for each point \( x \) in a dense set in \( M(A) \) the 0-germ is dense in the maximal ideal determined by \( x \). In other words the \( S \)-algebra condition is satisfied on at least a dense subset. McKissick [5] has proved the following.

**Lemma 4.** Let \( D \) be the open unit disk. Then there is a sequence \( \{a_k\} \) in \( D \), \( 0 < |a_k| \leq |a_{k+1}| \rightarrow 1 \), such that for any \( \varepsilon' > 0 \) there is a sequence \( \{J_k\} \) of open disks in \( D \) centered at \( \{a_k\} \) respectively such that:

1. \( \sum \delta(D_k) < \varepsilon' \).
2. There exist rational functions \( r_n \) with poles at \( a_1, \ldots, a_n \) such that \( r_n \rightarrow f \) uniformly on \( (\bigcup_{k=1}^{\infty} J_k)' \) and \( f = 0 \) on \( D' \) while \( f(0) = 1 \).

Using the above lemma we prove the following.

**Lemma 5.** Let \( c = |a_1|/2 \). There is a constant \( M > 0 \) such that for any positive \( \varepsilon \), \( \delta \) there is a \( \delta' \) and \( \{D_k\} \) a sequence of open disks in \( N(0, \delta'/c) - N(0, \delta c) \) such that:

1. \( \sum \delta(D_k) < \delta' c \).
2. There exist rational functions \( \{r_n\} \) with poles in \( D_1 \cup \cdots \cup D_n \) such that \( r_n \rightarrow g \) uniformly on \( (\bigcup_{k=1}^{\infty} D_k)' \) and
   (i) \( |g| \leq M \) on \( (\bigcup_{k=1}^{\infty} D_k)' \),
   (ii) \( g = 0 \) on \( N(0, \delta') \),
   (iii) \( |1 - g| < \varepsilon \) on \( N(0, \delta') \).

In fact if \( f \) is the function obtained by Lemma 1 with \( \varepsilon' \) a fixed constant (to be determined) independent of \( \varepsilon \) and \( \delta \), then \( \delta' \) can be chosen as \( \delta \delta(\varepsilon) \) where \( \delta(\varepsilon) \) is a function such that \( |z| < \delta(\varepsilon) \) implies \( |1 - f(z)| < \varepsilon \).

**Proof.** For disks \( \{J_k\} \) which we now choose in \( D \) let \( \{D_k\} \) be their respective images under the map \( 1/cz \). Since \( |a_k| \geq 2c \), by taking a sufficiently small \( \varepsilon' \) we can choose the open disks \( J_k \) guaranteed by Lemma 1.
so that \( z \in \bigcup J_k \) implies \( |z| > c \) and so that \( \sum \text{length}(\partial D_k) < 1 \). Thus \( D_k \subset N(0, 1/c^2) - N(0, 1) \) for all \( k \). Let \( f \) denote the limit on \((\bigcup J_k)'\) of the rational functions guaranteed by Lemma 1, and let \( M \) be the maximum of \( f \) on this set. Now since \( f(0) = 1, \ |z| < \delta(e) \) implies \( |1 - f(z)| < e \). Set \( \delta' = \delta(e) \) and let \( g(z) = f(\delta'/zc) \). Then redefining \( D_k \) as \( \delta'D_k \) we have \( D_k \subset N(0, \delta'/c^2) - N(0, \delta') \), \( g(z) \) is obviously defined for \( z \notin D_k \), and

(1) \[ \sum \text{length}(\partial D_k) < \delta', \]
(2) (i) \[ |g| < M \text{ on } (\bigcup D_k)', \]
(ii) \[ g(z) = 0 \text{ on } N(0, \delta'/c) \text{ since } |\delta'/zc| > 1 \text{ there, and} \]
(iii) \[ |1 - g(z)| < e \text{ on } N(0, \delta/c) \text{ since } |\delta'/zc| < \delta(e) \text{ there.} \]

The statement of the lemma follows by replacing \( \delta \) in the above by \( \delta c \).

**Corollary.** There is a constant \( M \) such that given positive \( \delta', e \) there exist \( D_k \) and \( g \) as in the above lemma satisfying (1) and (2) if \( \delta \) is taken as \( \delta'/\delta(e) \).

Of course the above lemmas hold with 0 replaced by any point \( p \). Also since the function \( f(z) = \sum \frac{1}{1|f'(a_k)(z-a_k)|} \) used by McKissick in Lemma 1 has a \( \delta(e) \beta e \) for some fixed \( \beta \) and small enough \( e \) we see that \( \delta(e) \) in the above statements can be replaced by \( e \). We now construct the example. Pick \( m > 1 \) such that \( 2^m e > 1 \). Let \( X_{m-1} = D \) and \( S_{m-1} = \phi \). Define \( S_n \subset X_n \), \( \{D_k^{(n)}\} \), for \( n \geq m \) inductively as follows. Suppose that \( S_{n-1} = \{a_1, \ldots, a_k\} \). Choose other points \( a_{k+1}, \ldots, a_l \) in \( X_{n-1} \) so that each point of \( X_{n-1} \) is within \( 1/2^n \) of some \( a_i \), and let \( S_n = \{a_1, \ldots, a_l\} \). Let \( d \) denote the minimum distance between the points of \( S_n \). Letting \( \delta = e = d/(2^n + c^{1/2}) \) find \( \{D_k^{(n)}\}_{k=1}^{\infty} \) open disks in \( N(a_j, \delta e/c) - N(a_j, \delta e c) \) such that \( \sum_{k=1}^{\infty} \text{length}(\partial D_k^{(n)}) < d^2/4^{n+i} < 1/2^{n+i} \) and (2) holds. Let \( X_n = X_{n-1} - \bigcup_{k,i} D_k^{(n)} \). Observe that since \( \delta e/c < d \) we have \( S_n \subset X_n \). Note too that \( \sum_{k=1}^{\infty} \text{length}(\partial D_k^{(n)}) < 1/2^n \) so that if we set \( X = \bigcap_{n=m}^{\infty} X_n \), we have excised a countable number of discs whose boundaries have total length \( < 1 \). Thus by Lemma 1 of [5], \( R(X) \subseteq C(X) \). It is now clear that given any \( e > 0 \) and any \( a_j \) some \( N(a_j, d/(2^n + c^{1/2})) \) is there is a \( g \) in \( R(X) \) so that \( \|g\| \leq M \), \( g \) vanishes on a neighborhood of \( a_j \), and \( |1 - g| \leq e \) on \( N(a_j, \delta') \). Thus the 0-germ at \( a_j \) is pointwise boundedly dense in the maximal ideal at \( a_j \) and so is dense. Since the \( \{a_j\} \) are a dense subset of \( X \) the example has the required properties.

Can the example be altered so that it is an \( S \)-algebra? One’s first inclination is to cover the disk by smaller and smaller \( \delta_n \) neighborhoods given by the Corollary, but clearly it is not possible to do this and even retain \( \sum \delta_n < \infty \). However the example is rather simple-minded in that the same function is used over and over. Perhaps a choice of other functions will extend the example. Some questions raised by the above are: (1) If the 0-germ at \( p \) is dense in the maximal ideal determined by \( p \), is \( p \) a peak.
point? (2) Is the example normal? (3) From an example of Cole (see also Basener [2]), it is well known that (iii) alone is not sufficient to imply the conclusion of the theorem. Are any of the hypotheses of the theorem redundant?

Wilken [8] has shown that if a uniform algebra $A$ is an $S$-algebra on $[0, 1]$ then $A = C[0, 1]$. In closing we also show the following.

**THEOREM.** If $A$ is a uniform algebra and $A$ is an $S$-algebra on the unit circle $T$, then $A = C(T)$.

**PROOF.** Let $p, q$ be peak points for $A$ in $T$, so $\{p, q\}$ is a peak set. Let $f$ in $A$ peak there. Then there are $g_n$ vanishing on neighborhoods of $p$ and $h_n$ vanishing on neighborhoods of $q$ such that $\|(1 - f^n)g_n\| < 1/n$ and $\|(1 - f^n)h_n\| < 1/n$ with $h_n$ and $g_n$ in $A$. Then $\|(1 - f^n)^2 - h_ng_n\| < 5/n$. Let $k_n = 0$ on one of the arcs $[p, q]$ joining $p$ to $q$ and let $k_n = h_ng_n$ on the other arc $[q, p]$. Then because $A$ is normal and hence local, $k_n$ are in $A$. But $k_n \to \chi_{(0,1)}$ boundedly pointwise. Thus if $\mu \in A^1$, $\mu_{[p, q]} = \mu_{[q, p]} \in A^1$. Hence $[p, q]$ is a peak set. Since every closed interval is an intersection of such peak sets, it follows that every closed set is a peak set and thus $A = C(T)$.

**BIBLIOGRAPHY**


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