Pin AND Pin' COBORDISM

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Abstract. The cobordism group \( \Omega^\text{Pin}_{*} \) of smooth manifolds with a Pin structure on the stable normal bundle is computed. The image \( \Omega^\text{Pin}_{*} \rightarrow \Omega_{*} \) is determined, and some generators for \( \Omega^\text{Pin}_{*} \) and \( \Omega_{*} \) Pin are given.

1. Notation. All manifolds will be smooth of class \( C^\infty \), and compact. The Lie group Pin\((n)\) is the double covering of \( O(n) \) whose identity component is Spin\((n)\). (See [3] for details.) Let \( O, SO, \text{Spin}, \) and Pin denote the usual stable groups. A bundle \( \xi \) has a Pin structure if \( w_2(\xi) + w_1(\xi) = 0 \). To see this consider the diagram of fibrations

\[
\begin{array}{ccc}
B \text{Pin}(1) & \rightarrow & B \text{Pin} \\
\downarrow & & \downarrow \\
BO(1) & \overset{i}{\rightarrow} & BO \\
\end{array}
\]

We need to find \( f^*(i) \). But since the identity component of \( B \text{Pin} \) is \( B \text{Spin} \), the fibration \( B \text{Pin}(1) \rightarrow BO(1) \) is nontrivial, hence \( i*f^*(i) \neq 0 \). But \( w_2 = 0 \) in \( BO(1) \), so \( a = 1 \). (This argument is due to R. E. Stong, and corrects the statement in [6].) A manifold is called a Pin manifold if its stable tangent bundle \( t \) has a Pin structure, and a Pin' manifold if its stable normal bundle \( v \) has a Pin structure. Pin and Pin' do not coincide, since \( w_2(t) = w_2(v) + w_1(v) \). For example, \( R\text{Pin}^n \) is a Pin' manifold, and \( R\text{Pin}^{n+2} \) is a Pin manifold.

In [2] \( \Omega^\text{Pin}_{*} \) was calculated by using the isomorphism \( \Omega^\text{Pin}_{*} \cong \Omega^\text{Spin}_{*}(RP^\infty) \). Analogously there is the following long exact sequence, due to Stong, relating Spin and Pin' cobordism, where \( \wedge \) is the cobordism theory introduced in [4]

\[
\cdots \rightarrow \Omega^\text{Spin}_{n} \rightarrow \wedge_n \rightarrow \Omega^\text{Pin}_{n-1} \rightarrow \Omega^\text{Pin}_{n-1} \rightarrow \cdots
\]

2. Algebraic methods. Let \( M \) be a closed manifold. Since the stable normal bundle \( v \) of \( M \) has a Pin structure iff \( w_2(v) + w_1(v) = 0 \), the classifying space \( B \text{Pin'} \) may be obtained from \( BO \) by killing \( w_2 + w_1 \), i.e., \( B \text{Pin'}(n) \) is the total space of the fibration over \( BO(n) \) induced from the path space

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over $K(Z_2, 2)$ via the map sending the fundamental class to $w_2 + w_2^2$, and $B \text{Pin'} = \lim B \text{Pin'}(n)$. A standard Serre spectral sequence argument shows the following.

**Lemma 2.1.**
(a) $H^*(B \text{Pin'}; Z_2) \cong H^*(BO; Z_2) / \text{Im}(H^*(K(Z_2; 2), Z_2)) \cong Z_2[w_i, i \neq 2^j + 1, i > 0, j \geq 0]$.  
(b) $H^*(B \text{Pin'}(n), k) \cong H^*(BO(n), k)$ if $k = Z_p$ or $Q$, for all $n$.

Let $\gamma_n$ be the pullback of the canonical bundle over $BO(n)$ to $B \text{Pin'}(n)$, and $M \text{Pin'}(n) = M(\gamma_n)$. Following Thom, there is a cofibration $BO(n) \to BO(n+1) \to MO(n+1)$. Lifting this over $B \text{Pin'}$ gives a cofibration $B \text{Pin'}(n) \to B \text{Pin'}(n+1) \to M \text{Pin'}(n+1)$. Then (b) implies $H^*(M \text{Pin'}(n)) \otimes k \cong H^*(MO(n)) \otimes k$ for $k = Z_p$ or $Q$. So $H^*(M \text{Pin'}) \otimes k = 0$, and hence all elements of $\Omega^*_\text{Pin'}$ have order a power of $2$. For the remainder of this note, all homology and cohomology will be with coefficients $Z_2$.

Since a Spin bundle has a Pin structure and a Pin' structure, there is a map $\alpha: B \text{Spin} \to B \text{Pin'}$ with $\alpha^*: H^*(B \text{Pin'}) \to H^*(B \text{Spin})$ given by $\alpha^*(w_i) = w_i$ if $i \neq 1$ and $\alpha^*(w_i) = 0$. So as a ring but not as an $A$ module

$$H^*(B \text{Pin'}) \cong H^*(B \text{Spin}) \otimes Z_2[w_1]$$

and via the Thom isomorphism

$$H^*(M \text{Pin'}) \cong H^*(M \text{Spin}) \otimes Z_2[w_1]$$

Also $\Omega^*_\text{Pin'}$ is an $\Omega^*_\text{Spin}$ module, since the product of a Spin manifold and a Pin' manifold is a Pin' manifold. Note that $B \text{Spin}$ is the sphere bundle of the line bundle $\xi$ over $B \text{Pin'}$ with $w_1(\xi) = w_1$.

We now define the building blocks of $H^*(M \text{Pin'})$ as an $A$ module. Let $W$, $\overline{W}$ be graded vector spaces over $Z_2$, with a basis for $W$ being $\{x_i\}, i \geq 0$, deg$(x_i) = 4i$ and a basis for $\overline{W}$ being $\{y_j, z_i, u_i, y_0\}, i \geq 0$, with deg$(y_j) = 4j$, deg$(z_i) = 4i + 2$, deg$(u_i) = 4i + 3$, deg$(y_0) = 1$. Let $M$ be the quotient of $A \otimes W$ by the relations $Sq^iSq^j(x_0) = 0, Sq^iSq^j(x_{i-1}) = Sq^j(x_i), i > 0$. Let $N$ be the quotient of $A \otimes \overline{W}$ by the relations $Sq^1(z_0) = Sq^2(y_0), Sq^1(z_{i+1}) = Sq^2Sq^3(z_i) + Sq^5Sq^3(u_i)$. Note that the submodules of $N$ generated by $y_i$ and $u_i$ are free.

For any finite sequence $J = (j_1, j_2, \cdots, j_r)$ of integers, let $n(J) = j_1 + \cdots + j_r$. Let $V$ be the graded $Z_2$ vector space with basis $\{v_J\}$, where $J$ runs over all finite sequences of integers $(j_1, \cdots, j_r)$ with $j_i > 1$ and $j_i \leq j_{i+1}$ for all $i$ and $n(J)$ is even. Define deg$(v_J) = 4n(J)$. Let $V'$ have basis $\{v_J\}$ with $J$ running through all finite sequences $(j_1, \cdots, j_r)$ with $j_i > 1$ and $j_i \leq j_{i+1}$ for all $i$, and $n(J)$ odd. Define deg$(v_J) = 4n(J) - 2$. Then from [1] we have that there is a vector space $\overline{V}$ such that $H^*(M \text{Spin}) \cong A / A(Sq^1, Sq^2 \otimes V' \oplus A(Sq^3) \otimes V' \oplus \overline{A} \otimes \overline{V'},$ and furthermore $\overline{V}$ has no elements of degree less than $20$. Let $\xi$ be an indeterminate of degree 1, and let $V^* = V \otimes Z_2[\xi]$. 


Theorem 2.1. As an $\mathcal{A}$ module $H^*(M \text{ Pin}')$ is isomorphic to 
\[
(M \otimes V) \oplus (N \otimes V') \oplus (\mathcal{A} \otimes V'').
\]

Proof. Let $P = M \otimes V \oplus N \otimes V' \oplus \mathcal{A} \otimes V''$. The map $\alpha^*: H^*(M \text{ Pin}') \to H^*(M \text{ Spin})$ has a $\mathbb{Z}_2$-module inverse $g$ with $g(w_{i_1} \cdots w_{i_k} U_{\text{Spin}}) = w_{i_1} \cdots w_{i_k} U_{\text{Pin'}}$, where $U$ denotes the Thom class. Let $\alpha_j = 1 \otimes v_j, \beta_j = 1 \otimes v_J', \gamma_j$ be the $\mathcal{A}$ module generators in $H^*(M \text{ Spin})$. Define $f: P \to H^*(M \text{ Pin}')$ by defining it as follows on the generators, and extending to an $\mathcal{A}$ map.

\[
\begin{align*}
 f(x_i \otimes v_J) &= w_i^{4t}g(\alpha_J), \\
 f(y_i \otimes v_J') &= w_i^{4t}g(\beta_J), \\
 f(z_i \otimes v_J') &= w_i^{4t}Sq^8 g(\beta_J), \\
 f(u_i \otimes v_J) &= w_i^{4t+3}g(\beta_J), \\
 f(\varepsilon_i \otimes v_J) &= w_i^2g(\gamma_i), \text{ where } t_i \text{ runs through a basis of } \mathcal{V}.
\end{align*}
\]

It is routine to check that the map is well defined on $P$. For example, if $g(\beta_J) = p_J U$, then $Sq^1 p_J = 0 = Sq^2 p_J$, so $Sq^3 Sq^1 (p_J U) = 0$. Also we have $Sq^3 (g(\beta_J)) = Sq^3 (w_1 g(\beta_J)) + w_2^3 g(\beta_J)$, and therefore for any integer $j \geq 0$ and for any sequence $I$, $w_i^t Sq^I (g(\alpha_I))$, $w_i^t Sq^I (g(\beta_I))$ and $w_i^t Sq^I (g(\gamma_I))$ are in the image of $f$. Since the $\{\alpha_J, \beta_J, \gamma_J\}$ form an $\mathcal{A}$ basis for $H^*(M \text{ Spin})$, and $H^*(M \text{ Pin}') \approx H^*(M \text{ Spin}) \otimes \mathbb{Z}_2[w_1]$ as vector spaces, $f$ must be onto. A counting argument then shows $f$ must be an isomorphism.

The next step is to compute the $E_2$ term of the Adams spectral sequence. This is $\text{Ext}_{\mathcal{A}}(H^*(M \text{ Pin}'), Z_2) = \text{Ext}_{\mathcal{A}}(M \otimes V; Z_2) \oplus \text{Ext}_{\mathcal{A}}(N \otimes V'; Z_2) \oplus \text{Ext}_{\mathcal{A}}(\mathcal{A} \otimes V''; Z_2) = \text{Ext}_{\mathcal{A}}(M; Z_2) \otimes V \oplus \text{Ext}_{\mathcal{A}}(N; Z_2) \otimes V' \oplus V''$. But $M \approx (M \otimes_{\mathcal{A}} 1) \otimes_{\mathcal{A}} 1$ and $N \approx (N \otimes_{\mathcal{A}} 1) \otimes_{\mathcal{A}} 1$ since the only relations

\[
\begin{align*}
\text{Ext}_{\mathcal{A}}(M, Z_2) &= \begin{cases}
8 & \text{if } t \equiv s \\
7 & \text{if } t \not\equiv s
\end{cases} \\
\end{align*}
\]
in $M$ and $N$ come from $\mathcal{A}_1$ ($\mathcal{A}_1$ is the subalgebra of $\mathcal{A}$ generated by 1, $Sq^1$, and $Sq^2$). Hence we need only compute $\text{Ext}^*_{\mathcal{A}_1}(M \otimes_{\mathcal{A}_1} Z_2)$ and $\text{Ext}^*_{\mathcal{A}_1}(N \otimes_{\mathcal{A}_1} Z_2)$. Since $\mathcal{A}_1$ is six dimensional, one can do this as in [4] or via a filtration. Figures 1 and 2 give a picture of the answer in low dimensions.

Here vertical lines denote multiplication by $h_0$ and diagonal lines multiplication by $h_1$, and multiplication by $\omega$ corresponds to multiplication by $[\omega] \in \Omega^\text{spin}_8$ (see [2]). $\text{Ext}^*_M(M, Z_2) = 0$ unless $t-s \equiv 0, 2, 3, 4 \text{ mod } 8$, and $\text{Ext}^*_N(N, Z_2) = 0$ for $t-s \not\equiv 0, 2, 3, 4 \text{ mod } 8$, with $s>0$.

**Lemma 2.2.** The Adams spectral sequence for $\Omega^\text{Pin}^*$ collapses. Furthermore all extensions, except those given by multiplication by $h_0$ are trivial.

**Proof.** A theorem of Margolis asserts that if an $\mathcal{A}$ module $Q$ is $H^*(X)$ for some $X$, and if $Q \cong Q' \oplus F$ where $F$ is a free $\mathcal{A}$ module, then $Q$ is $H^*(X' \vee K)$ where $K$ is an Eilenberg-Mac Lane object, and $H^*(X') \cong Q'$. Hence the free summands in $H^*(M \text{ Pin}')$ cannot give rise to nonzero differentials, nor nontrivial extensions. Since the differentials must commute with multiplication by $h_1$, they are all 0. The only possible nontrivial extensions (other than multiplication by $h_0$) are in dimensions congruent to 2 or 3 mod 8.

But $\Omega^\text{Pin}^*$ is an $\Omega^\text{spin}_8$ module and the only possible nontrivial extensions in dimensions congruent to 2 mod 8 are $[\omega^r]$ and $[\omega^s]$ where $[\omega] \in \Omega^\text{spin}_8$. By induction $[r]$ and $[s]$ have order 2. Since the elements in dim $8k+3$ which might possibly support nontrivial extensions are just $h_1$ times the elements in $8k+2$, their extensions must also be trivial.

$\Omega^\text{Pin}^*$ is just the sum of the contributions from each summand. The
contribution from \( M \otimes v_f \) in dimension \( 4n(J) + 8k + l \) is \( \mathbb{Z}_2 \) if \( l = 2, 3 \); \( \mathbb{Z}_{2k+1} \) if \( l = 0 \) and \( \mathbb{Z}_{2k+4} \) if \( l = 4 \), and 0 otherwise. The contribution from \( N \otimes v_f' \) in dimension \( 4n(J) - 2 + 8k + l \) is \( \mathbb{Z}_2 \) if \( l = 1, 3, 4, 7 \); \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) if \( l = 0, k > 0 \); \( \mathbb{Z}_2 \) if \( l = 0, k = 0 \); \( \mathbb{Z}_{2k+2} \) if \( l = 2 \); \( \mathbb{Z}_{2k+3} \) if \( l = 6 \), and 0 otherwise. Thus we have the following table in low dimensions.

<table>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
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<td>( \mathbb{Z}_2 )</td>
<td>( \mathbb{Z}_2 )</td>
<td>( \mathbb{Z}_2 )</td>
<td>( \mathbb{Z}_2 )</td>
<td>0</td>
<td>0</td>
<td>( \mathbb{Z}_2^2 )</td>
<td>0</td>
<td>( \mathbb{Z}_2^2 )</td>
<td>( \mathbb{Z}_2^2 )</td>
<td>( \mathbb{Z}_2^2 \otimes \mathbb{Z}_2^2 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Geometrical considerations. Recall that in [4] a cobordism theory \( \Lambda_\ast \) was introduced, consisting of pairs \( (M, \xi) \), where \( \xi \) is a line bundle over the oriented manifold \( M \), and \( w_2(M) = w_1(\xi)^2 \). The following theorem is due to Stong.

**Theorem 3.1.** The sequence

\[
\cdots \to \Omega_n^{\text{Spin}} \xrightarrow{\beta} \Lambda_n \xrightarrow{\gamma} \Omega_{n-1}^{\text{Pin}} \xrightarrow{\delta} \Omega_{n-1}^{\text{Spin}} \to \cdots
\]

is exact, where \( \beta[M] = [M, 1], \gamma[M, \xi] = \text{the class of the submanifold of } M \text{ dual to } \xi, \) and \( \delta(N) = \text{the class of the oriented double cover of } N. \)

**Proof.** Let \( E_n \) be the total space of the fibration induced from the path fibration by the map \( f: BSO(n) \times BO(1) \to K(Z_2, 2) \) with \( f^*(l) = w_2 \otimes 1 + l \otimes w_1^3 \). Denote the projection \( E_n \to BSO(n) \times BO(1) \) by \( p \). Let \( \gamma_n \) and \( \xi \) be the pullbacks to \( E_n \) of the canonical bundles over \( BSO(n) \) and \( BO(1) \). Then as in [4], \( \Lambda_\ast = \lim_n \pi_{n+k}(M(\gamma_n)) \). Let \( X_n \) be the disc bundles of the pullback of \( \xi \) to \( D(\gamma_n) \) and let \( Y_n \) be the disc bundle of the pullback of \( \xi \) to \( S(\gamma_n) \). Then \( Y_n \subset X_n \), and \( \partial X_n \) and \( \partial Y_n \) are the corresponding sphere bundles. Hence in the stable range there is an exact sequence

\[
\cdots \to \pi_{n+k}(\partial X_n/\partial Y_n) \to \pi_{n+k}(X_n/Y_n) \to \pi_{n+k}(X_n/Y_n \cup \partial X_n) \to \pi_{n+k-1}(\partial X_n/\partial Y_n) \to \cdots
\]

(This makes sense since everything is \( (n-1) \) connected.) But \( S(\xi) = B\text{ Spin}(n) \), and hence \( \pi_r(\partial X_n/\partial Y_n) = \pi_r(M(\gamma_n)) \). Similarly \( \pi_r(X/Y) \approx \pi_r(M(\gamma_n)) \). Thus we have two of the three groups identified. Now \( X_n/Y_n \cup \partial X_n \sim M(\gamma_n \otimes \xi) \). So we need to identify \( \lim_n \pi_{n+k}(M(\gamma_n \otimes \xi)) \) with \( \Omega_{n-1}^{\text{Pin}} \).

But there is a commutative diagram

\[
E_n \longrightarrow B\text{ Pin}(n + 1) \\
\downarrow \quad \downarrow \\
BSO(n) \times BO(1) \longrightarrow BO(n + 1),
\]
where the bottom map classifies the product of the canonical bundles, and both horizontal maps are homotopy equivalences up to dimension \( n \). The top map is covered by a bundle isomorphism between \( \gamma_n + \xi \) and the canonical bundle over \( B \text{Pin}'(n+1) \). This identifies the three groups. The geometric description of the maps is a simple exercise for the reader.

We now need two lemmas which were essentially proved in [4].

**Lemma 3.2.** The kernel of \( \beta \) is the set of those classes in \( \Omega_*^{\text{Spin}} \) which can be written as \([M] \cdot [S^1]^2\), where \( M \) is any element of \( \Omega_*^{\text{Spin}} \) and \( S^1 \) is the circle with the nonbounding Spin structure.

**Lemma 3.3.** The element of order \( 2^{2n+2} \) in \( \wedge_{4n+1} \) is \([RP^{4n+1}, \eta]\), where \( \eta \) is the Hopf bundle.

**Theorem 3.4.** (a) In \( \Omega_{4n}^{\text{Pin}}[RP^{4n}] \) has order \( 2^{2n+2} \) if \( n \) is odd, and \( 2^{2n+1} \) if \( n \) is even.

(b) In \( \Omega_{4n+2}^{\text{Pin}}[RP^{4n+2}] \) has order \( 2^{2n+3} \) if \( n \) is even, and \( 2^{2n+2} \) if \( n \) is odd.

**Proof.** Use 3.1. \( \beta[RP^{4n+1}, \eta] = [RP^{4n}] \). Since \([RP^{4n}]\) has filtration 0 in \( \Omega_{4n}^{\text{Pin}} \), and \([RP^{4n+1}, \eta]\) has order \( 2^{2n+2} \), \([RP^{4n}]\) must have order at least \( 2^{2n+1} \). But there is only one such element, and it has the required order. This proves (a). Let \( \Delta : \Omega_{4n}^{\text{Pin}} \to \Omega_{n-2}^{\text{Pin}} \) be given by \( \Delta(M) = \{ \text{the submanifold dual to } \zeta \oplus \zeta \text{ with the inherited Pin structure} \} \) where \( \zeta \) is the orientation line bundle. \( \Delta \) is a well-defined homomorphism. Note that \( \Delta([RP^{4n}]) = [RP^{4n-2}] \). Similarly define \( \Delta' : \Omega_{n}^{\text{Pin}} \to \Omega_{n-2}^{\text{Pin}} \). Then \( \Delta([RP^{4n+2}]) = [RP^{4n}] \).

Now \([RP^{4(n+1)}]\) has order \( 2^{2n+4} \) if \( n \) is even, and \( \Delta' \Delta([RP^{4(n+1)}]) = [RP^{4n}] \), which has order \( 2^{2n+1} \). Thus \([RP^{4n+2}]\) must have order at least \( 2^{2n+1} \) and it must have filtration 0. By 2 there is only one such element, and it must have the correct order.

**Corollary 3.5.** The subgroup of \( \Omega_{4n}^{\text{Pin}} \) generated by elements of order greater than two is just the subgroup generated by the products \( [M_i] \times [RP^{4n}] \) where \( M_i \) is the generator element in \( \Omega_*^{\text{Spin}} \) described in 1. The same is true for Pin with \( RP^{4n} \) replaced by \( RP^{4n+2} \).

**Theorem 3.6.** The image of \( f_* : \Omega_*^{\text{Pin}} \to \mathfrak{N}_* \) is the set of cobordism classes all of whose Stiefel-Whitney numbers involving \( w_2 + w_1^2 \) vanish.

**Proof.** Since the Adams spectral sequence for \( \Omega_*^{\text{Pin}} \) is trivial, and \( f^* : H^*(MO) \to H^*(M \text{Pin}') \) is onto, Theorem 6.2 of [2] implies that \( \text{Im}(\Omega_{n}^{\text{Pin}} \to \mathfrak{N}_n) = \{ g \in \pi_n(MO) \mid g^*(\text{Ker } f^*) = 0 \in H^*(s^n) \} \).

The argument on p. 468 of [2] shows that this set is precisely those classes whose numbers divisible by \( w_1^2 + w_2 \) vanish.
REFERENCES


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