Pin AND Pin' COBORDISM

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Abstract. The cobordism group $\Omega_\text{Pin}^n$ of smooth manifolds with a Pin structure on the stable normal bundle is computed. The image $\Omega_\text{Pin}^n \rightarrow \Omega_\text{Pin'}^n$ is determined, and some generators for $\Omega_\text{Pin}^n$ and $\Omega_\text{Pin'}^n$ are given.

1. Notation. All manifolds will be smooth of class $C^\infty$, and compact. The Lie group Pin$(n)$ is the double covering of $O(n)$ whose identity component is Spin$(n)$. (See [3] for details.) Let $O$, $SO$, Spin, and Pin denote the usual stable groups. A bundle $\xi$ has a Pin structure if $w_2(\xi) + w_1(\xi) = 0$. To see this consider the diagram of fibrations

$$
B\text{Pin}(1) \rightarrow B\text{Pin} \rightarrow BO(1) \rightarrow BO \rightarrow K(\mathbb{Z}_2, 2).
$$

We need to find $f^*(i)$. But since the identity component of $B$ Pin is $B$ Spin, $f^*(i) = w_2 + aw_1^2$ where $a \in \mathbb{Z}_2$. Since Pin$(1) = \mathbb{Z}_4$, the fibration $B\text{Pin}(1) \rightarrow BO(1)$ is nontrivial, hence $i*f^*(i) \neq 0$. But $w_2 = 0$ in $BO(1)$, so $a = 1$. (This argument is due to R. E. Stong, and corrects the statement in [6].) A manifold is called a Pin manifold if its stable tangent bundle $t$ has a Pin structure, and a Pin' manifold if its stable normal bundle $v$ has a Pin structure. Pin and Pin' do not coincide, since $w_2(t) = w_2(v) + w_1(v)$. For example, $R\text{Pin}^n$ is a Pin' manifold, and $R\text{Pin'}^{n+2}$ is a Pin manifold.

In [2] $\Omega_\text{Pin}^n$ was calculated by using the isomorphism $\Omega_\text{Pin}^n \cong \Omega_\text{Spin}^n(R\mathbb{P}^\infty)$. Analogously there is the following long exact sequence, due to Stong, relating Spin and Pin' cobordism, where $\wedge$ is the cobordism theory introduced in [4]

$$
\cdots \rightarrow \Omega_\text{Spin}^n \rightarrow \wedge_n \rightarrow \Omega_\text{Pin'}^n \rightarrow \Omega_\text{Spin}^{n-1} \rightarrow \cdots.
$$

2. Algebraic methods. Let $M$ be a closed manifold. Since the stable normal bundle $v$ of $M$ has a Pin structure if $w_2(v) + w_1^2(v) = 0$, the classifying space $B$ Pin' may be obtained from $BO$ by killing $w_2 + w_1^2$, i.e., $B$ Pin'(n) is the total space of the fibration over $BO(n)$ induced from the path space.

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over $K(\mathbb{Z}_2, 2)$ via the map sending the fundamental class to $w_2 + w_2^2$, and $B \text{Pin'} = \lim B \text{Pin'}(n)$. A standard Serre spectral sequence argument shows the following.

**Lemma 2.1.** (a) $H^*(B \text{Pin'}; \mathbb{Z}_2) \approx H^*(BO; \mathbb{Z}_2) / \text{Im}(H^*(K(\mathbb{Z}_2; 2), \mathbb{Z}_2)) \approx \mathbb{Z}_2\langle w_i, i \neq 2^j + 1, i > 0, j \geq 0 \rangle$.

(b) $H^*(B \text{Pin'}(n), k) \approx H^*(BO(n), k)$ if $k = \mathbb{Z}_2$ or $\mathbb{Q}$, for all $n$.

Let $\gamma_n$ be the pullback of the canonical bundle over $BO(n)$ to $B \text{Pin'}(n)$, and $M \text{Pin'}(n) = M(\gamma_n)$. Following Thom, there is a cofibration $BO(n) \to BO(n+1) \to MO(n+1)$. Lifting this over $B \text{Pin'}$ gives a cofibration $B \text{Pin'}(n) \to B \text{Pin'}(n+1) \to M \text{Pin'}(n+1)$. Then (b) implies $H^*(M \text{Pin'}(n)) \otimes k \approx H^*(MO(n)) \otimes k$ for $k = \mathbb{Z}_2$ or $\mathbb{Q}$. So $H^*(M \text{Pin'}) \otimes k = 0$, and hence all elements of $\Omega^\infty_{\text{Pin'}}$ have order a power of 2. For the remainder of this note, all homology and cohomology will be with coefficients $\mathbb{Z}_2$.

Since a Spin bundle has a Pin structure and a Pin' structure, there is a map $\alpha: B \text{Spin} \to B \text{Pin'}$ with $\alpha^*: H^*(B \text{Pin'}) \to H^*(B \text{Spin})$ given by $\alpha^*(w_i) = w_i$ if $i \neq 1$ and $\alpha^*(w_1) = 0$. So as a ring but not as an $\mathcal{A}$ module $H^*(B \text{Pin'}) \approx H^*(B \text{Spin}) \otimes \mathbb{Z}_2\langle w_1 \rangle$

and via the Thom isomorphism

$H^*(M \text{Pin'}) \approx H^*(M \text{Spin}) \otimes \mathbb{Z}_2\langle w_1 \rangle$ as $\mathbb{Z}_2$ vector spaces.

Also $\Omega^\infty_{\text{Pin'}}$ is an $\Omega^\infty_{\text{Spin}}$ module, since the product of a Spin manifold and a Pin' manifold is a Pin' manifold. Note that $B \text{Spin}$ is the sphere bundle of the line bundle $L$ over $B \text{Pin'}$ with $wx = w$.

We now define the building blocks of $H^*(M \text{Pin'})$ as an $\mathcal{A}$ module. Let $W, \overline{W}$ be graded vector spaces over $\mathbb{Z}_2$, with a basis for $W$ being $\{x_i\}$, $i \geq 0$, $\deg(x_i) = 4i$ and a basis for $\overline{W}$ being $\{y_i, z_i, u_i, y_0\}$, $i \geq 0$, with $\deg(y_i) = 4i$, $\deg(z_i) = 4i + 2$, $\deg(u_i) = 4i + 3$, $\deg(y_0) = 1$. Let $M$ be the quotient of $\mathcal{A} \otimes W$ by the relations $S^2q^2x_0 = 0$, $S^2q^2(x_{i-1}) = S^2(x_i)$, $i > 0$. Let $N$ be the quotient of $\mathcal{A} \otimes \overline{W}$ by the relations $S^2q^2(y_0) = S^2q^2(y_i)$, $S^2q^2(z_{i+1}) = S^2q^2(z_i) + S^2q^2(u_i)$. Note that the submodules of $N$ generated by $y_i$ and $u_i$ are free.

For any finite sequence $J = (j_1, j_2, \cdots, j_r)$ of integers, let $n(J) = j_1 + \cdots + j_r$. Let $V$ be the graded $\mathbb{Z}_2$ vector space with basis $\{v_J\}$, where $J$ runs over all finite sequences of integers $(j_1, \cdots, j_r)$ with $j_i > 1$ and $j_i \leq j_{i+1}$ for all $i$ and $n(J)$ is even. Define $\deg(v_J) = 4n(J)$. Let $V'$ have basis $\{v_J\}$ with $J$ running through all finite sequences $(j_1, \cdots, j_r)$ with $j_i > 1$ and $j_i \leq j_{i+1}$ for all $i$, and $n(J)$ odd. Define $\deg(v_J) = 4n(J) - 2$. Then from [1] we have that there is a vector space $\tilde{V}$ such that $H^*(M \text{Spin}) \approx \mathcal{A}/\mathcal{A}(S^1, S^2) \otimes V \oplus \mathcal{A}/\mathcal{A}(S^2) \otimes V' \oplus \mathcal{A} \otimes \overline{V}$, and furthermore $\tilde{V}$ has no elements of degree less than 20. Let $\xi$ be an indeterminate of degree 1, and let $V'' = V \otimes \mathbb{Z}_2[\xi]$. 
Theorem 2.1. As an $\mathcal{A}$ module $H^*(M \text{Pin}')$ is isomorphic to

$$(M \otimes V) \oplus (N \otimes V') \oplus (\mathcal{A} \otimes V'').$$

Proof. Let $P = M \otimes V \oplus N \otimes V' \oplus \mathcal{A} \otimes V''$. The map $\alpha^*: H^*(M \text{Pin}') \to H^*(M \text{Spin})$ has a $\mathbb{Z}_2$-module inverse $g$ with $g(w_{i_1} \cdots w_{i_k} U_{\text{Spin}}) = w_{i_1} \cdots w_{i_k} U_{\text{Pin}}$, where $U$ denotes the Thom class. Let $\alpha_j = 1 \otimes v_j, \beta_j = 1 \otimes v_j, \gamma_i$ be the $\mathcal{A}$ module generators in $H^*(M \text{Spin})$. Define $f: P \to H^*(M \text{Pin}')$ by defining it as follows on the generators, and extending to an $\mathcal{A}$ map.

$$
\begin{align*}
 f(x_i \otimes v_j) &= w_i^4 g(\alpha_j), \\
 f(y_i \otimes v'_j) &= w_i^4 g(\beta_j), \\
 f(z_i \otimes v'_j) &= w_i^4 Sq^8 g(\beta_j), \\
 f(u_i \otimes v'_j) &= w_i^4+3 g(\beta_j), \\
 f(1 \otimes \xi^i t_i) &= w_i^4 g(\gamma_i),
\end{align*}
$$

where $t_i$ runs through a basis of $V$.

It is routine to check that the map is well defined on $P$. For example, if $g(\beta_j) = p_j U$, then $Sq^1 p_j = 0 = Sq^2 p_j$, so $Sq^1 Sq^1 (p_j U) = Sq^2 (p_j w_1 U) = 0$. Also we have $Sq^2(g(\beta_j)) = Sq^1(w_1 g(\beta_j)) + w_1^2 g(\beta_j)$, and therefore for any integer $j \geq 0$ and for any sequence $I$, $w_i^1 Sq^I (g(\alpha_j))$, $w_i^1 Sq^I (g(\beta_j))$ and $w_1^1 Sq^I (g(\gamma_j))$ are in the image of $f$. Since the $\alpha_j, \beta_j, \gamma_i$ form an $\mathcal{A}$ basis for $H^*(M \text{Spin})$, and $H^*(M \text{Pin}') \cong H^*(M \text{Spin}) \otimes \mathbb{Z}_2[w_1]$ as vector spaces, $f$ must be onto. A counting argument then shows $f$ must be an isomorphism.

The next step is to compute the $E_2$ term of the Adams spectral sequence. This is $\text{Ext}_{\mathcal{A}}(H^*(M \text{Pin}'), \mathbb{Z}_2) = \text{Ext}_{\mathcal{A}}(M \otimes V; \mathbb{Z}_2) \oplus \text{Ext}_{\mathcal{A}}(N \otimes V'; \mathbb{Z}_2) \oplus \text{Ext}_{\mathcal{A}}(\mathcal{A} \otimes V''; \mathbb{Z}_2) = \text{Ext}_{\mathcal{A}}(M; \mathbb{Z}_2) \otimes V \oplus \text{Ext}_{\mathcal{A}}(N; \mathbb{Z}_2) \otimes V' \oplus V''$. But $M \cong (M \otimes \mathcal{A}_1) \otimes \mathcal{A}_1 \otimes \mathcal{A}$ and $N \cong (N \otimes \mathcal{A}_1) \otimes \mathcal{A}_1 \otimes \mathcal{A}$ since the only relations

![Figure 1](image-url)
in $M$ and $N$ come from $\mathcal{A}_1$ ($\mathcal{A}_1$ is the subalgebra of $\mathcal{A}$ generated by 1, $Sq^1$, and $Sq^2$). Hence we need only compute $\text{Ext}^A_\mathcal{A}(M \otimes_{\mathcal{A}_1} \mathcal{A}_1; \mathbb{Z}_2)$ and $\text{Ext}^A_\mathcal{A}(N \otimes_{\mathcal{A}_1} \mathcal{A}_1; \mathbb{Z}_2)$. Since $\mathcal{A}_1$ is six dimensional, one can do this as in [4] or via a filtration. Figures 1 and 2 give a picture of the answer in low dimensions.

Here vertical lines denote multiplication by $h_0$ and diagonal lines multiplication by $h_1$, and multiplication by $\omega$ corresponds to multiplication by $[\omega] \in \Omega^\text{spin}_8$ (see [2]). $\text{Ext}^A_\mathcal{A}(M, \mathbb{Z}_2) = 0$ unless $t-s \equiv 0, 2, 3, 4 \mod 8$, and $\text{Ext}^A_\mathcal{A}(N, \mathbb{Z}_2) = 0$ for $t-s \not\equiv 0, 2, 3, 4 \mod 8$, with $s > 0$.

**Lemma 2.2.** The Adams spectral sequence for $\Omega^n_{\text{Pin}}$ collapses. Furthermore all extensions, except those given by multiplication by $h_0$ are trivial.

**Proof.** A theorem of Margolis asserts that if an $\mathcal{A}$ module $Q$ is $H^*(X)$ for some $X$, and if $Q \cong Q' \oplus F$ where $F$ is a free $\mathcal{A}$ module, then $Q$ is $H^*(X' \vee K)$ where $K$ is an Eilenberg-Mac Lane object, and $H^*(X') \cong Q'$. Hence the free summands in $H^*(M \text{Pin}')$ cannot give rise to nonzero differentials, nor nontrivial extensions. Since the differentials must commute with multiplication by $h_1$, they are all 0. The only possible nontrivial extensions (other than multiplication by $h_0$) are in dimensions congruent to 2 or 3 mod 8.

But $\Omega^n_{\text{Pin}}$ is an $\Omega^\text{spin}_8$ module and the only possible nontrivial extensions in dimensions congruent to 2 mod 8 are $[\omega]^r$ and $[\omega]^s$ where $[\omega] \in \Omega^\text{spin}_8$. By induction $[r]$ and $[s]$ have order 2. Since the elements in dim $8k+3$ which might possibly support nontrivial extensions are just $h_1$ times the elements in $8k+2$, their extensions must also be trivial.

$\Omega^n_{\text{Pin}}$ is just the sum of the contributions from each summand. The
contribution from $M \otimes v_f$ in dimension $4n(J) + 8k + l$ is $\mathbb{Z}_2$ if $l = 2, 3$; $\mathbb{Z}_{2k+1}$ if $l = 0$ and $\mathbb{Z}_{2k+4}$ if $l = 4$, and 0 otherwise. The contribution from $N \otimes v_f'$ in dimension $4n(J) - 2 + 8k + l$ is $\mathbb{Z}_2$ if $l = 1, 3, 4, 7$; $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if $l = 0$, $k > 0$, $\mathbb{Z}_2$ if $l = 0, k = 0$, $\mathbb{Z}_{2k+2}$ if $l = 2$, $\mathbb{Z}_{2k+3}$ if $l = 6$, and 0 otherwise. Thus we have the following table in low dimensions.

<table>
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<th>0</th>
<th>1</th>
<th>2</th>
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<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$\mathbb{Z}_2$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
<td>$\mathbb{Z}_4$</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
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<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
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3. Geometrical considerations. Recall that in [4] a cobordism theory $\Lambda_*$ was introduced, consisting of pairs $(M, \xi)$, where $\xi$ is a line bundle over the oriented manifold $M$, and $w_2(M) = w_1(\xi)^2$. The following theorem is due to Stong.

**Theorem.** 3.1. The sequence

$$
\cdots \to \Omega_n^{Spin} \xrightarrow{\beta} \Lambda_n \xrightarrow{\gamma} \Omega_{n-1}^{Pin'} \xrightarrow{\delta} \Omega_{n-1}^{Spin} \to \cdots
$$

is exact, where $\beta[M] = [M, 1]$, $\gamma[M, \xi] = \text{the class of the submanifold of } M \text{ dual to } \xi$, and $\delta(N)$ is the class of the oriented double cover of $N$.

**Proof.** Let $E_n$ be the total spaces of the fibration induced from the path fibration by the map $f: BSO(n) \times BO(1) \to K(\mathbb{Z}_2, 2)$ with $f^*(i) = w_2 \otimes 1 + 1 \otimes w_1^2$. Denote the projection $E_n \to BSO(n) \times BO(1)$ by $p$. Let $\gamma_n$ and $\xi$ be the pullbacks to $E_n$ of the canonical bundles over $BSO(n)$ and $BO(1)$. Then as in [4], $\Lambda_k = \lim_n \pi_{n+k}(M(\gamma_n))$. Let $X_n$ be the disc bundles of the pullback of $\xi$ to $D(\gamma_n)$ and let $Y_n$ be the disc bundle of the pullback of $\xi$ to $S(\gamma_n)$. Then $Y_n \subset X_n$, and $\partial X_n$ and $\partial Y_n$ are the corresponding sphere bundles.

Hence in the stable range there is an exact sequence

$$
\cdots \to \pi_{n+k}(\partial X_n/\partial Y_n) \to \pi_{n+k}(X_n/Y_n) \\
\to \pi_{n+k}(X_n/Y_n \cup \partial X_n) \to \pi_{n+k-1}(\partial X_n/\partial Y_n) \to \cdots
$$

(This makes sense since everything is $(n-1)$ connected.) But $S(\xi) = BSpin(n)$, and hence $\pi_i(\partial X_n/\partial Y_n) = \pi_i(MSpin(n))$. Similarly $\pi_i(X/Y) \approx \pi_i(M(\gamma_n))$. Thus we have two of the three groups identified. Now $X_n/Y_n \cup \partial X_n \sim M(\gamma_n \oplus \xi)$. So we need to identify $\lim_n \pi_{n+k}(M(\gamma_n \oplus \xi))$ with $\Omega_{n}^{Pin'}$. But there is a commutative diagram

$$
E_n \to BPin'(n + 1) \\
\downarrow \quad \downarrow \\
BSO(n) \times BO(1) \to BO(n + 1),
$$
where the bottom map classifies the product of the canonical bundles, and both horizontal maps are homotopy equivalences up to dimension \( n \). The top map is covered by a bundle isomorphism between \( \gamma_n + \xi \) and the canonical bundle over \( B \text{Pin}'(n+1) \). This identifies the three groups. The geometric description of the maps is a simple exercise for the reader.

We now need two lemmas which were essentially proved in [4].

**Lemma 3.2.** The kernel of \( \beta \) is the set of those classes in \( \Omega_{\text{Spin}}^* \) which can be written as \( [M] \cdot [S^1]^2 \), where \( M \) is any element of \( \Omega_{\text{Spin}}^* \) and \( S^1 \) is the circle with the nonbounding Spin structure.

**Lemma 3.3.** The element of order \( 2^{2n+2} \) in \( \wedge_{4n+1} \) is \( [\text{RP}^{4n+1}, \eta] \), where \( \eta \) is the Hopf bundle.

**Theorem 3.4.** (a) In \( \Omega_{4n}^\text{Pin'}[\text{RP}^{4n}] \) has order \( 2^{2n+2} \) if \( n \) is odd, and \( 2^{2n+1} \) if \( n \) is even.

(b) In \( \Omega_{4n+2}^\text{Pin}[\text{RP}^{4n+2}] \) has order \( 2^{2n+3} \) if \( n \) is even, and \( 2^{2n+2} \) if \( n \) is odd.

**Proof.** Use 3.1. \( \beta[\text{RP}^{4n+1}, \eta] = [\text{RP}^{4n}] \). Since \( [\text{RP}^{4n}] \) has filtration 0 in \( \Omega_{4n}^\text{Pin'} \), and \( [\text{RP}^{4n+1}, \eta] \) has order \( 2^{2n+2} \), \( [\text{RP}^{4n}] \) must have order at least \( 2^{2n+1} \). But there is only one such element, and it has the required order. This proves (a). Let \( \Delta : \Omega_{4n}^\text{Pin'} \to \Omega_{4n-2}^\text{Pin} \) be given by \( \Delta(M) = \{ \text{the submanifold dual to } \xi \oplus \zeta \text{ with the inherited Pin structure} \} \) where \( \xi \) is the orientation line bundle. \( \Delta \) is a well-defined homomorphism. Note that \( \Delta ([\text{RP}^{4n}]) = [\text{RP}^{4n-2}] \). Similarly define \( \Delta' : \Omega_{n}^\text{Pin} \to \Omega_{n-2}^\text{Pin} \). Then \( \Delta([\text{RP}^{4n+2}]) = [\text{RP}^{4n}] \). Now \( [\text{RP}^{4(n+1)}] \) has order \( 2^{2n+4} \) if \( n \) is even, and \( \Delta'( [\text{RP}^{4(n+1)}] ) = [\text{RP}^{4n}] \), which has order \( 2^{2n+1} \). Thus \( [\text{RP}^{4n+2}] \) must have order at least \( 2^{2n+1} \) and it must have filtration 0. By 2 there is only one such element, and it must have the correct order.

**Corollary 3.5.** The subgroup of \( \Omega_{4n}^\text{Pin}' \) generated by elements of order greater than two is just the subgroup generated by the products \( [M_j] \times [\text{RP}^{4n}] \) where \( M_j \) is the generator element in \( \Omega_{\text{Spin}}^* \) described in 1. The same is true for \( \text{Pin} \) with \( \text{RP}^{4n} \) replaced by \( \text{RP}^{4n+2} \).

**Theorem 3.6.** The image of \( f_* : \Omega_{\text{Spin}}^* \to \mathcal{N}_* \) is the set of cobordism classes all of whose Stiefel-Whitney numbers involving \( w_2 + w_1^2 \) vanish.

**Proof.** Since the Adams spectral sequence for \( \Omega_{\text{Spin}}^* \) is trivial, and \( f^* : H^*(M) \to H^*(M \text{Pin}') \) is onto, Theorem 6.2 of [2] implies that

\[
\text{Im}(\Omega_{n}^\text{Pin} \to \mathcal{N}_n) = \{ g \in \pi_n(M) \mid g^*(\text{Ker } f^*) = 0 \in H^*(S^n) \}.
\]

The argument on p. 468 of [2] shows that this set is precisely those classes whose numbers divisible by \( w_2^2 + w_2 \) vanish.
REFERENCES


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