PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 2, July 1973

Pin AND Pin′ COBORDISM

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ABSTRACT. The cobordism group $\Omega_*^{\mathrm{Pin}'}$ of smooth manifolds with a Pin structure on the stable normal bundle is computed. The image $\Omega_*^{\mathrm{Pin}'}\to\mathfrak{N}_*$ is determined, and some generators for Ω_*^{Pin} and $\Omega_*^{\mathrm{Pin}'}$ are given.

1. **Notation.** All manifolds will be smooth of class C^∞, and compact. The Lie group $\mathrm{Pin}(n)$ is the double covering of $O(n)$ whose identity component is $\mathrm{Spin}(n)$. (See **[3]** for details.) Let O, SO, Spin, and Pin denote the usual stable groups. A bundle ξ has a Pin structure if $w_2(\xi)+w_1^2(\xi)=0$. To see this consider the diagram of fibrations

$$\begin{array}{ccccc} B\,\mathrm{Pin}(1) & \to & B\,\mathrm{Pin} & & \\ \downarrow & & \downarrow & & \\ BO(1) & \xrightarrow{i} & BO & \xrightarrow{f} & K(Z_2, 2). \end{array}$$

We need to find $f^*(\iota)$. But since the identity component of B Pin is B Spin, $f^*(\iota)=w_2+aw_1^2$ where $a\in Z_2$. Since $\mathrm{Pin}(1)=Z_4$, the fibration $B\,\mathrm{Pin}(1)\to BO(1)$ is nontrivial, hence $i^*f^*(\iota)\neq 0$. But $w_2=0$ in $BO(1)$, so $a=1$. (This argument is due to R. E. Stong, and corrects the statement in **[6]**.) A manifold is called a Pin mainfold if its stable tangent bundle τ has a Pin structure, and a Pin′ manifold if its stable normal bundle ν has a Pin structure. Pin and Pin′ do not coincide, since $w_2(\tau)=w_2(\nu)+w_1^2(\nu)$. For example, $\boldsymbol{RP}^{4n}$ is a Pin′ manifold, and $\boldsymbol{RP}^{4n+2}$ is a Pin manifold.

In **[2]** Ω_*^{Pin} was calulated by using the isomorphism $\Omega_*^{\mathrm{Pin}}\approx\Omega_*^{\mathrm{Spin}}(\boldsymbol{RP}^\infty)$. Analogously there is the following long exact sequence, due to Stong, relating Spin and Pin′ cobordism, where $\bigwedge$ is the cobordism theory introduced in **[4]**

$$\cdots\to\Omega_n^{\mathrm{Spin}}\to\bigwedge\nolimits_n\to\Omega_{n-1}^{\mathrm{Pin}'}\to\Omega_{n-1}^{\mathrm{Spin}}\to\cdots.$$

2. **Algebraic methods.** Let M be a closed manifold. Since the stable normal bundle ν of M has a Pin structure iff $w_2(\nu)+w_1^2(\nu)=0$, the classifying space $B\,\mathrm{Pin}'$ may be obtained from BO by killing $w_2+w_1^2$, i.e., $B\,\mathrm{Pin}'(n)$ is the total space of the fibration over $BO(n)$ induced from the path space

Received by the editors August 8, 1972 and, in revised form, September 27, 1972.
AMS (MOS) subject classifications (1970). Primary 57D90; Secondary 55G10, 55H15.
Key words and phrases. Pin cobordism.

over $K(Z_2, 2)$ via the map sending the fundamental class to $w_2+w_1^2$, and $B\,\mathrm{Pin}'=\lim B\,\mathrm{Pin}'(n)$. A standard Serre spectral sequence argument shows the following.

LEMMA 2.1. (a) $H^*(B\,\mathrm{Pin}'; Z_2)\approx H^*(BO; Z_2)/\!\!/\mathrm{Im}(H^*(K(Z_2; 2), Z_2))\approx Z_2[w_i, i\neq 2^j+1, i>0, j\geqq 0]$.

(b) $H^*(B\,\mathrm{Pin}'(n), k)\approx H^*(BO(n), k)$ *if* $k=Z_p$ *or* Q, *for all* n.

Let γ_n be the pullback of the canonical bundle over $BO(n)$ to $B\,\mathrm{Pin}'(n)$, and $M\,\mathrm{Pin}'(n)=M(\gamma_n)$. Following Thom, there is a cofibration $BO(n)\to BO(n+1)\to MO(n+1)$. Lifting this over $B\,\mathrm{Pin}'$ gives a cofibration $B\,\mathrm{Pin}'(n)\to B\,\mathrm{Pin}'(n+1)\to M\,\mathrm{Pin}'(n+1)$. Then (b) implies $H^*(M\,\mathrm{Pin}'(n))\otimes k\approx H^*(MO(n))\otimes k$ for $k=Z_p$ or Q. So $H^*(M\,\mathrm{Pin}')\otimes k=0$, and hence all elements of $\Omega_*^{\mathrm{Pin}'}$ have order a power of 2. For the remainder of this note, all homology and cohomology will be with coefficients Z_2.

Since a Spin bundle has a Pin structure and a Pin′ structure, there is a map $\alpha: B\,\mathrm{Spin}\to B\,\mathrm{Pin}'$ with $\alpha^*: H^*(B\,\mathrm{Pin}')\to H^*(B\,\mathrm{Spin})$ given by $\alpha^*(w_i)=w_i$ if $i\neq 1$ and $\alpha^*(w_1)=0$. So as a ring but not as an $\mathscr{A}$ module

$$H^*(B\,\mathrm{Pin}') \approx H^*(B\,\mathrm{Spin}) \otimes Z_2[w_1]$$

and via the Thom isomorphism

$$H^*(M\,\mathrm{Pin}') \approx H^*(M\,\mathrm{Spin}) \otimes Z_2[w_1] \text{ as } Z_2 \text{ vector spaces.}$$

Also $\Omega_*^{\mathrm{Pin}'}$ is an Ω_*^{Spin} module, since the product of a Spin manifold and a Pin′ manifold is a Pin′ manifold. Note that $B\,\mathrm{Spin}$ is the sphere bundle of the line bundle ξ over $B\,\mathrm{Pin}'$ with $w_1(\xi)=w_1$.

We now define the building blocks of $H^*(M\,\mathrm{Pin}')$ as an $\mathscr{A}$ module. Let W, $\overline{W}$ be graded vector spaces over Z_2, with a basis for W being $\{x_i\}$, $i\geqq 0$, $\deg(x_i)=4i$ and a basis for $\overline{W}$ being $\{y_i, z_i, u_i, y_0'\}$, $i\geqq 0$, with $\deg(y_i)=4i$, $\deg(z_i)=4i+2$, $\deg(u_i)=4i+3$, $\deg(y_0')=1$. Let M be the quotient of $\mathscr{A}\otimes W$ by the relations $Sq^2Sq^1(x_0)=0$, $Sq^2Sq^3(x_{i-1})=Sq^1(x_i)$, $i>0$. Let N be the quotient of $\mathscr{A}\otimes\overline{W}$ by the relations $Sq^1(z_0)=Sq^2(y_0')$, $Sq^1(z_{i+1})=Sq^2Sq^3(z_i)+Sq^2Sq^2(u_i)$. Note that the submodules of N generated by y_i and u_i are free.

For any finite sequence $J=(j_1, j_2, \cdots, j_r)$ of integers, let $n(J)=j_1+\cdots+j_r$. Let V be the graded Z_2 vector space with basis $\{v_J\}$, where J runs over all finite sequences of integers $(j_1, \cdots, j_r)$ with $j_i>1$ and $j_i\leqq j_{i+1}$ for all i and $n(J)$ is even. Define $\deg(v_J)=4n(J)$. Let V' have basis $\{v_J'\}$ with J running through all finite sequences $(j_1, \cdots, j_r)$ with $j_i>1$ and $j_i\leqq j_{i+1}$ for all i, and $n(J)$ odd. Define $\deg(v_J')=4n(J)-2$. Then from **[1]** we have that there is a vector space $\bar{V}$ such that $H^*(M\,\mathrm{Spin})\approx\mathscr{A}/\mathscr{A}(Sq^1, Sq^2)\otimes V\oplus\mathscr{A}/\mathscr{A}(Sq^3)\otimes V'\oplus\mathscr{A}\otimes\bar{V}$, and furthermore $\bar{V}$ has no elements of degree less than 20. Let ξ be an indeterminate of degree 1, and let $V''=V\otimes Z_2[\xi]$.

THEOREM 2.1. *As an $\mathscr{A}$ module $H^*(M\,\mathrm{Pin}')$ is isomorphic to*

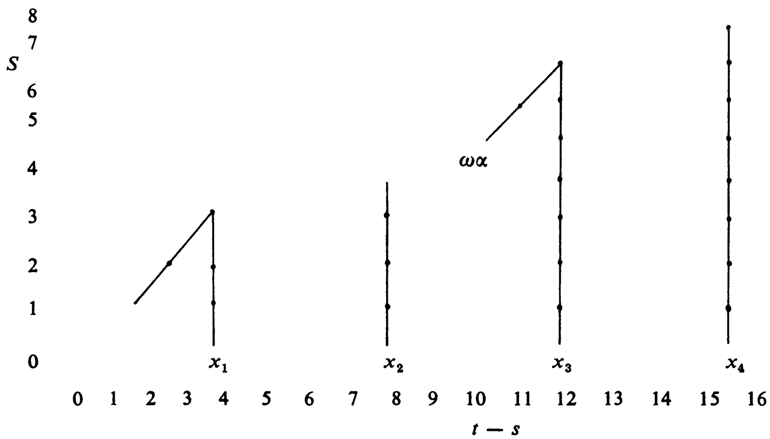
$$(M \otimes V) \oplus (N \otimes V') \oplus (\mathscr{A} \otimes V'').$$

PROOF. Let $P = M\otimes V \oplus N\otimes V' \oplus \mathscr{A}\otimes V''$. The map $\alpha^*: H^*(M\,\mathrm{Pin}') \to H^*(M\,\mathrm{Spin})$ has a Z_2-module inverse g with $g(w_{i_1} \cdots w_{i_k} U_{\mathrm{Spin}}) = w_{i_1} \cdots w_{i_k} U_{\mathrm{Pin}'}$, where U denotes the Thom class. Let $\alpha_J = 1 \otimes v_J$, $\beta_J = 1 \otimes v'_J$, γ_i be the $\mathscr{A}$ module generators in $H^*(M\,\mathrm{Spin})$. Define $f: P \to H^*(M\,\mathrm{Pin}')$ by defining it as follows on the generators, and extending to an $\mathscr{A}$ map.

$$f(x_i \otimes v_J) = w_1^{4i} g(\alpha_J),$$
$$f(y_i \otimes v'_J) = w_1^{4i} g(\beta_J), \quad f(y'_0 \otimes v'_J) = w_1 g(\beta_J),$$
$$f(z_i \otimes v'_J) = w_1^{4i} Sq^2 g(\beta_J), \quad f(u_i \otimes v'_J) = w_1^{4i+3} g(\beta_J),$$
$$f(1 \otimes \xi^k t_i) = w_1^k g(\gamma_i), \quad \text{where } t_i \text{ runs through a basis of } \bar{V}.$$

It is routine to check that the map is well defined on P. For example, if $g(\beta_J) = p_J U$, then $Sq^1 p_J = 0 = Sq^2 p_J$, so $Sq^2 Sq^1(p_J U) = Sq^2(p_J w_1 U) = 0$. Also we have $Sq^2(g(\beta_J)) = Sq^1(w_1 g(\beta_J)) + w_1^2 g(\beta_J)$, and therefore for any integer $j \geqq 0$ and for any sequence I, $w_1^j Sq^I(g(\alpha_I))$, $w_1^j Sq^I(g(\beta_J))$ and $w_1^j Sq^I(g(\gamma_i))$ are in the image of f. Since the $\{\alpha_J, \beta_J, \gamma_i\}$ form an $\mathscr{A}$ basis for $H^*(M\,\mathrm{Spin})$, and $H^*(M\,\mathrm{Pin}') \approx H^*(M\,\mathrm{Spin}) \otimes Z_2[w_1]$ as vector spaces, f must be onto. A counting argument then shows f must be an isomorphism.

The next step is to compute the E_2 term of the Adams spectral sequence. This is $\mathrm{Ext}_{\mathscr{A}}(H^*(M\,\mathrm{Pin}'), Z_2) = \mathrm{Ext}_{\mathscr{A}}(M \otimes V; Z_2) \oplus \mathrm{Ext}_{\mathscr{A}}(N \otimes V'; Z_2) \oplus \mathrm{Ext}_{\mathscr{A}}(\mathscr{A} \otimes V''; Z_2) = \mathrm{Ext}_{\mathscr{A}}(M; Z_2) \otimes V \oplus \mathrm{Ext}_{\mathscr{A}}(N; Z_2) \otimes V' \oplus V''$. But $M \approx (M \otimes_{\mathscr{A}} \mathscr{A}_1) \otimes_{\mathscr{A}_1} \mathscr{A}$ and $N \approx (N \otimes_{\mathscr{A}} \mathscr{A}_1) \otimes_{\mathscr{A}_1} \mathscr{A}$ since the only relations



$\mathrm{Ext}_{\mathscr{A}}^{s,t}(M, Z_2)$

FIGURE 1

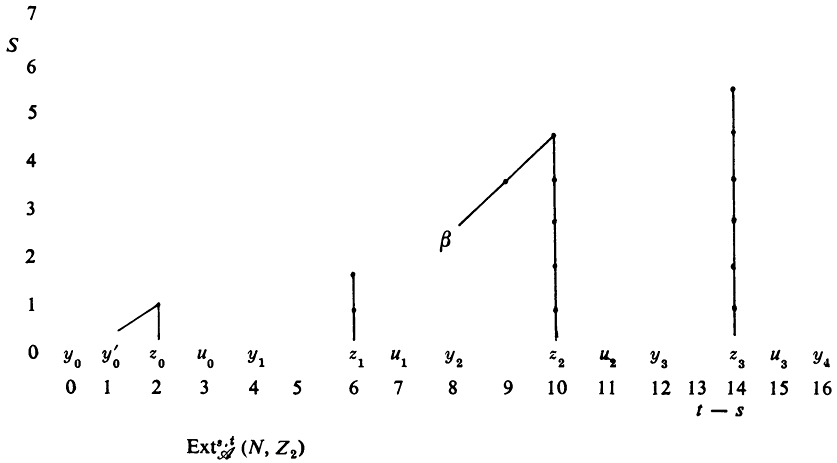


FIGURE 2

in M and N come from $\mathscr{A}_1$ ($\mathscr{A}_1$ is the subalgebra of $\mathscr{A}$ generated by 1, Sq^1, and Sq^2). Hence we need only compute $\mathrm{Ext}_{\mathscr{A}_1}(M \otimes_{\mathscr{A}} \mathscr{A}_1; Z_2)$ and $\mathrm{Ext}_{\mathscr{A}_1}(N \otimes_{\mathscr{A}} \mathscr{A}_1; Z_2)$. Since $\mathscr{A}_1$ is six dimensional, one can do this as in [**4**] or via a filtration. Figures 1 and 2 give a picture of the answer in low dimensions.

Here vertical lines denote multiplication by h_0 and diagonal lines multiplication by h_1, and multiplication by ω corresponds to multiplication by $[\omega] \in \Omega_8^{\mathrm{Spin}}$ (see [**2**]). $\mathrm{Ext}_{\mathscr{A}}^{s,t}(M, Z_2)=0$ unless $t-s\equiv 0, 2, 3, 4 \bmod 8$, and $\mathrm{Ext}_{\mathscr{A}}^{s,t}(N, Z_2)=0$ for $t-s\not\equiv 0, 2, 3, 4 \bmod 8$, with $s>0$.

LEMMA 2.2. *The Adams spectral sequence for $\Omega_*^{\mathrm{Pin}'}$ collapses. Furthermore all extensions, except those given by multiplication by h_0 are trivial.*

PROOF. A theorem of Margolis asserts that if an $\mathscr{A}$ module Q is $H^*(X)$ for some X, and if $Q\approx Q'\oplus F$ where F is a free $\mathscr{A}$ module, then Q is $H^*(X'\vee K)$ where K is an Eilenberg-Mac Lane object, and $H^*(X')\approx Q'$. Hence the free summands in $H^*(M\,\mathrm{Pin}')$ cannot give rise to nonzero differentials, nor nontrivial extensions. Since the differentials must commute with multiplication by h_1, they are all 0. The only possible nontrivial extensions (other than multiplication by h_0) are in dimensions congruent to 2 or 3 mod 8.

But $\Omega_*^{\mathrm{Pin}'}$ is an Ω_*^{Spin} module and the only possible nontrivial extensions in dimensions congruent to 2 mod 8 are $[\omega^k r]$ and $[\omega^l s]$ where $[\omega] \in \Omega_8^{\mathrm{Spin}}$. By induction $[r]$ and $[s]$ have order 2. Since the elements in dim $8k+3$ which might possibly support nontrivial extensions are just h_1 times the elements in $8k+2$, their extensions must also be trivial.

$\Omega_n^{\mathrm{Pin}'}$ is just the sum of the contributions from each summand. The contribution from $M \otimes v_J$ in dimension $4n(J)+8k+l$ is Z_2 if $l=2$, 3; $Z_{2^{4k+1}}$ if $l=0$ and $Z_{2^{4k+4}}$ if $l=4$, and 0 otherwise. The contribution from $N \otimes v'_J$ in dimension $4n(J)-2+8k+l$ is Z_2 if $l=1$, 3, 4, 7; $Z_2 \oplus Z_2$ if $l=0$, $k>0$, Z_2 if $l=0$, $k=0$, $Z_{2^{4k+2}}$ if $l=2$, $Z_{2^{4k+3}}$ if $l=6$, and 0 otherwise. Thus we have the following table in low dimensions.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\Omega_n^{\mathrm{Pin}'}$	Z_2	0	Z_2	Z_2	Z_{2^4}	0	0	0	Z_{2^5}	0	Z_2^2	Z_2^2	$Z_2^8 \oplus Z_{2^2}$

3. **Geometrical considerations.** Recall that in **[4]** a cobordism theory Λ_* was introduced, consisting of pairs (M, ξ), where ξ is a line bundle over the oriented manifold M, and $w_2(M)=w_1(\xi)^2$. The following theorem is due to Stong.

THEOREM. 3.1. *The sequence*

$$\cdots \longrightarrow \Omega_n^{\mathrm{Spin}} \xrightarrow{\beta} \Lambda_n \xrightarrow{\gamma} \Omega_{n-1}^{\mathrm{Pin}'} \xrightarrow{\delta} \Omega_{n-1}^{\mathrm{Spin}} \longrightarrow \cdots$$

is exact, where $\beta[M]=[M, 1]$, $\gamma[M, \xi]=$ *the class of the submanifold of* M *dual to* ξ, *and* $\delta(N)$ *is the class of the oriented double cover of* N.

PROOF. Let E_n be the total spaces of the fibration induced from the path fibration by the map $f: BSO(n) \times BO(1) \to K(Z_2, 2)$ with $f^*(i)=w_2 \otimes 1+1 \otimes w_1^2$. Denote the projection $E_n \to BSO(n) \times BO(1)$ by p. Let γ_n and ξ be the pullbacks to E_n of the canonical bundles over $BSO(n)$ and $BO(1)$. Then as in **[4]**, $\Lambda_k = \lim_n \pi_{n+k}(M(\gamma_n))$. Let X_n be the disc bundles of the pullback of ξ to $D(\gamma_n)$ and let Y_n be the disc bundle of the pullback of ξ to $S(\gamma_n)$. Then $Y_n \subset X_n$, and ∂X_n and ∂Y_n are the corresponding sphere bundles. Hence in the stable range there is an exact sequence

$$\cdots \to \pi_{n+k}(\partial X_n / \partial Y_n) \to \pi_{n+k}(X_n / Y_n)$$
$$\to \pi_{n+k}(X_n / Y_n \cup \partial X_n) \to \pi_{n+k-1}(\partial X_n / \partial Y_n) \to \cdots.$$

(This makes sense since everything is $(n-1)$ connected.) But $S(\xi)= B\,\mathrm{Spin}(n)$, and hence $\pi_r(\partial X_n/\partial Y_n)=\pi_r(M\,\mathrm{Spin}(n))$. Similarly $\pi_r(X/Y) \approx \pi_r(M(\gamma_n))$. Thus we have two of the three groups identified. Now $X_n/Y_n \cup \partial X_n \sim M(\gamma_n \oplus \xi)$. So we need to identify $\lim_n \pi_{n+k}(M(\gamma_n \oplus \xi))$ with $\Omega_{k-1}^{\mathrm{Pin}'}$. But there is a commutative diagram

$$\begin{array}{ccc} E_n & \longrightarrow & B\,\mathrm{Pin}'(n+1) \\ \downarrow & & \downarrow \\ BSO(n) \times BO(1) & \to & BO(n+1), \end{array}$$

where the bottom map classifies the product of the canonical bundles, and both horizontal maps are homotopy equivalences up to dimension n. The top map is covered by a bundle isomorphism between $\gamma_n+\xi$ and the canonical bundle over $B\,\mathrm{Pin}'(n+1)$. This identifies the three groups. The geometric description of the maps is a simple exercise for the reader.

We now need two lemmas which were essentially proved in [4].

LEMMA 3.2. *The kernel of* β *is the set of those classes in* Ω_*^{Spin} *which can be written as* $[M]\cdot[S^1]^2$, *where* M *is any element of* Ω_*^{Spin} *and* S^1 *is the circle with the nonbounding* Spin *structure.*

LEMMA 3.3. *The element of order* 2^{2n+2} *in* $\bigwedge_{4n+1}$ *is* $[RP^{4n+1},\eta]$, *where* η *is the Hopf bundle.*

THEOREM 3.4. (a) *In* $\Omega_{4n}^{\mathrm{Pin}'}[RP^{4n}]$ *has order* 2^{2n+2} *if* n *is odd, and* 2^{2n+1} *if* n *is even.*

(b) *In* $\Omega_{4n+2}^{\mathrm{Pin}}[RP^{4n+2}]$ *has order* 2^{2n+3} *if* n *is even, and* 2^{2n+2} *if* n *is odd.*

PROOF. Use 3.1. $\beta[RP^{4n+1},\eta]=[RP^{4n}]$. Since $[RP^{4n}]$ has filtration 0 in $\Omega_{4n}^{\mathrm{Pin}'}$, and $[RP^{4n+1},\eta]$ has order 2^{2n+2}, $[RP^{4n}]$ must have order at least 2^{2n+1}. But there is only one such element, and it has the required order. This proves (a). Let $\Delta:\Omega_n^{\mathrm{Pin}'}\to\Omega_{n-2}^{\mathrm{Pin}}$ be given by $\Delta(M)=\{$the submanifold dual to $\zeta\oplus\zeta$ with the inherited Pin structure$\}$ where ζ is the orientation line bundle. Δ is a well-defined homomorphism. Note that $\Delta([RP^{4n}])=[RP^{4n-2}]$. Similarly define $\Delta':\Omega_n^{\mathrm{Pin}}\to\Omega_{n-2}^{\mathrm{Pin}'}$. Then $\Delta([RP^{4n+2}])=[RP^{4n}]$. Now $[RP^{4(n+1)}]$ has order 2^{2n+4} if n is even, and $\Delta'\Delta([RP^{(4n+1)}])=[RP^{4n}]$, which has order 2^{2n+1}. Thus $[RP^{4n+2}]$ must have order at least 2^{2n+1} and it must have filtration 0. By 2 there is only one such element, and it must have the correct order.

COROLLARY 3.5. *The subgroup of* $\Omega_*^{\mathrm{Pin}'}$ *generated by elements of order greater than two is just the subgroup generated by the products* $[M_J]\times[RP^{4n}]$ *where* M_J *is the generator element in* Ω_*^{Spin} *described in* 1. *The same is true for Pin with* RP^{4n} *replaced by* RP^{4n+2}.

THEOREM 3.6. *The image of* $f_*:\Omega_*^{\mathrm{Pin}'}\to\mathfrak{N}_*$ *is the set of cobordism classes all of whose Stiefel-Whitney numbers involving* $w_2+w_1^2$ *vanish.*

PROOF. Since the Adams spectral sequence for $\Omega_*^{\mathrm{Pin}'}$ is trivial, and $f^*:H^*(MO)\to H^*(M\,\mathrm{Pin}')$ is onto, Theorem 6.2 of [2] implies that

$$\mathrm{Im}(\Omega_n^{\mathrm{Pin}'}\to\mathfrak{N}_n)=\{g\in\pi_n(MO)\mid g^*(\mathrm{Ker}\,f^*)=0\in H^*(s^n)\}.$$

The argument on p. 468 of [2] shows that this set is precisely those classes whose numbers divisible by $w_1^2+w_2$ vanish.

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