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ON THE REDUCTION OF COMPLEX BORDISM TO UNORIENTED BORDISM[1]

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ABSTRACT. The image of the natural transformation from the complex bordism of a CW complex X, $MU_*(X)$, to its unoriented bordism, $N_*(X)$, is contained in a subgroup identified with $H_*(X;Z)\otimes(N_*)^2$. A characterization is given for the CW complexes for which the image and the subgroup coincide.

Introduction. Let $MU_*(X)$ and $N_*(X)$ denote, respectively, the complex and the unoriented bordism of a CW complex X. For finite complexes, there are dual multiplicative cohomology theories $MU^*(\_)$ and $N^*(\_)$. There are natural forgetful transformations

$$\phi_*(X):MU_*(X)\to N_*(X) \quad \text{and} \quad \phi^*(X):MU^*(X)\to N^*(X).$$

Let $j:(N^*)^2\to N^*$ denote the inclusion of the subring consisting of squares of elements of the coefficient ring of unoriented (co)-bordism, $N^*\cong N_{-*}$. Recall that N_* has the structure of a graded Z_2 polynomial ring; so j is a split monomorphism. Boardman and Quillen have given a natural multiplicative isomorphism of cohomology theories,

$$\theta^*(X):N^*(X)\to H^*(X;Z_2)\otimes_{Z_2} N^*$$

([1], [6]). We have then a natural transformation of functors (the first functor does not give a cohomology theory).

$$\begin{aligned}\psi^*(X):H^*(X;Z)\otimes(N^*)^2 &\xrightarrow{\rho\otimes 1} H^*(X;Z_2)\otimes(N^*)^2 \cong H^*(X;Z_2)\otimes_{Z_2}(N^*)^2\\ &\xrightarrow{1\otimes j} H^*(X;Z_2)\otimes_{Z_2} N^* \xleftarrow{\theta^*(X)} N^*(X).\end{aligned}$$

Here ρ is reduction from integral to mod 2 cohomology. $\rho\otimes 1$ is monic as it appears as a factor of the following universal-coefficient-theorem monomorphism

$$H^*(X;Z)\otimes(N^*)^2 \xrightarrow{\rho\otimes 1} H^*(X;Z_2)\otimes(N^*)^2 \cong H^*(X;(N^*)^2).$$

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So $\psi^*(X)$ gives a multiplicative, monic, natural transformation of functors. Dually, there is a natural monomorphism of functors

$$\psi_*(X): H_*(X; Z) \otimes (N_*)^2 \to N_*(X).$$

Recall that $Z_{(2)}$ is the integers localized at the prime two (the subring of rationals represented by fractions with odd denominators). $Z_{(2)}$ is a flat abelian group and $MU_*(\_\_)\otimes Z_{(2)}$ is a homology theory with an external multiplication.

Both homomorphisms $\phi_*(X)$ and $\psi_*(X)$ have range $N_*(X)$. When X is a one point space, Milnor showed that their images coincide [**4**]. The purpose of this note is to record the following observation which provides a converse to and includes the result of [**5**].

THEOREM. *For any* CW *complex* X, image $\phi_*(X) \subset$ image $\psi_*(X)$, image $\phi_*(X) =$ image $\psi_*(X)$ *if and only if the projective dimension of* $MU(X)\otimes Z_{(2)}$ *as a* $MU_*\otimes Z_{(2)}$ *module is at most one.*

Of course, for finite complexes, the analogous statement relating the images of $\phi^*(X)$ and $\psi^*(X)$ is also true (but the reader should be warned that the projective dimensions of $MU_*(X)\otimes Z_{(2)}$ and of $MU^*(X)\otimes Z_{(2)}$ as $MU_*\otimes Z_{(2)} \cong MU^{-*}\otimes Z_{(2)}$ modules are not in general equal). Since $\psi^*(X)$ is monic and multiplicative, we obtain the following corollary which may be of interest.

COROLLARY. *For finite complexes, there is a natural multiplicative transformation of functors*

$$\psi^*(X)^{-1} \circ \phi^*(X): MU^*(X) \to H^*(X; Z) \otimes (N^*)^2.$$

$\psi^*(X)^{-1} \circ \phi^*(X)$ is epic if and only if the projective dimension of $MU^*(X)\otimes Z_{(2)}$ as a $MU^*\otimes Z_{(2)}$ module is at most one. □

PROOF OF THEOREM. Given a CW complex X, there is a -1-connected CW spectrum A with $H_*(A; Z)$ free abelian and a stable map $f: A \to X$ such that the induced homomorphism in complex bordism (f_3 in diagram (*))

$$(*) \qquad \begin{tikzcd}
 & 0 \arrow[d] & \\
H_*(A; Z) \otimes (N_*)^2 \arrow[r, "f_1"] \arrow[d, "\psi_*(A)"] & H_*(X; Z) \otimes (N_*)^2 \arrow[d, "\psi_*(X)"] & \\
N_*(A) \arrow[r, "f_2"] & N_*(X) & \\
MU_*(A) \arrow[r, "f_3"] \arrow[u, "\phi_*(A)"] & MU_*(X) \arrow[u, "\phi_*(X)"] \arrow[r] & 0
\end{tikzcd}$$

is epic ([2, Proposition 2.4 for the finite case]; [3, Lemma 5 for the generalization]).

From Milnor's result that image $\psi_*(\text{point})=\text{image } \phi_*(\text{point})$ and from the fact that $H_*(A;Z)$ is free abelian, it is routine to show that the images of $\psi_*(A)$ and $\phi_*(A)$ coincide. From the commutativity of $(*)$, we have image $\phi_*(X)=\phi_*(X)\circ f_3=f_2(\text{image } \phi_*(A))=f_2(\text{image } \psi_*(A))=$ image $\psi_*(X)\circ f_1\subseteq \text{image } \psi_*(X)$. Since $\psi_*(X)$ is a monomorphism, image $\phi_*(X)=\text{image } \psi_*(X)\circ f_1$ is precisely image $\psi_*(X)$ if and only if $f_1=H_*(f;Z)\otimes(N_*)^2$ is epic. We may identify Z_2 as the zero component of the graded Z_2 vector space, $(N_*)^2$. We have a chain of equivalences:

$f_1=H_*(f;Z)\otimes(N_*)^2$ is epic
$\Leftrightarrow H_*(f,Z)\otimes Z_2$ is epic
$\Leftrightarrow f_4=H_*(f,Z_{(2)})\cong H_*(f;Z)\otimes Z_{(2)}$ is epic
$\Leftrightarrow \mu(X)\otimes Z_{(2)}$ is epic in $(**)$
$\Leftrightarrow$ the projective dimension of $MU_*(X)\otimes Z_{(2)}$ as a $MU_*\otimes Z_{(2)}$ module is at most one.

The first two of these equivalences follow from elementary algebra. The third is by consideration of the commutative diagram $(**)$.

$$(**)\qquad \begin{array}{ccccc} MU_*(A)\otimes Z_{(2)} & \xrightarrow{f_3\otimes Z_{(2)}} & MU_*(X)\otimes Z_{(2)} & \to & 0 \\ \downarrow{\scriptstyle \mu(A)\otimes Z_{(2)}} & & \downarrow{\scriptstyle \mu(X)\otimes Z_{(2)}} & & \\ H_*(A;Z_{(2)}) & \xrightarrow{f_4} & H_*(X;Z_{(2)}) & & \\ \downarrow & & & & \\ 0 & & & & \end{array}$$

Since $H_*(A;Z)$ is free abelian, the Thom homomorphism $\mu(A)$ is epic. The last equivalence is by Corollary 3.11 $\otimes Z_{(2)}$ of [3] as extended in [2]. $\square$

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