ON THE REDUCTION OF COMPLEX BORDISM TO UNORIENTED BORDISM

DAVID COPELAND JOHNSON

Abstract. The image of the natural transformation from the complex bordism of a CW complex $X$, $MU_*(X)$, to its unoriented bordism, $N_*(X)$, is contained in a subgroup identified with $H_*(X; \mathbb{Z}) \otimes (N_*)^2$. A characterization is given for the CW complexes for which the image and the subgroup coincide.

Introduction. Let $MU_*(X)$ and $N_*(X)$ denote, respectively, the complex and the unoriented bordism of a CW complex $X$. For finite complexes, there are dual multiplicative cohomology theories $MU^*(-)$ and $N^*(-)$. There are natural forgetful transformations

$$\phi_*: MU_*(X) \to N_*(X) \quad \text{and} \quad \phi^*(X): MU^*(X) \to N^*(X).$$

Let $y:(N^*)^2 \to N^*$ denote the inclusion of the subring consisting of squares of elements of the coefficient ring of unoriented (co)-bordism, $N^* \subset N_-^*$. Recall that $N_*$ has the structure of a graded $\mathbb{Z}_2$ polynomial ring; so $j$ is a split monomorphism. Boardman and Quillen have given a natural multiplicative isomorphism of cohomology theories,

$$\theta^*(X): N^*(X) \to H^*(X; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} N^*$$

([1, 6]). We have then a natural transformation of functors (the first functor does not give a cohomology theory).

$$\psi^*(X): H^*(X; \mathbb{Z}) \otimes (N^*)^2 \xrightarrow{\rho \otimes 1} H^*(X; \mathbb{Z}_2) \otimes (N^*)^2 \cong H^*(X; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} (N^*)^2 \xrightarrow{1 \otimes j} H^*(X; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} N^* \xrightarrow{\theta^*(X)} N^*(X).$$

Here $\rho$ is reduction from integral to mod 2 cohomology. $\rho \otimes 1$ is monic as it appears as a factor of the following universal-coefficient-theorem monomorphism

$$H^*(X; Z) \otimes (N^*)^2 \xrightarrow{\rho \otimes 1} H^*(X; \mathbb{Z}_2) \otimes (N^*)^2 \cong H^*(X; (N^*)^2).$$

Received by the editors November 1, 1972.


Key words and phrases. Complex bordism, unoriented bordism, projective dimension.

1 Supported by NSF GP-33883.
So \( \psi^*(X) \) gives a multiplicative, monic, natural transformation of functors.
Dually, there is a natural monomorphism of functors
\[
\psi_*(X) : H_*(X; Z) \otimes (N_*)^2 \rightarrow N_*(X).
\]
Recall that \( Z_{(2)} \) is the integers localized at the prime two (the subring of rationals represented by fractions with odd denominators). \( Z_{(2)} \) is a flat abelian group and \( MU_*(\_ ) \otimes Z_{(2)} \) is a homology theory with an external multiplication.

Both homomorphisms \( \phi_*(X) \) and \( \psi_*(X) \) have range \( N_*(X) \). When \( X \) is a one point space, Milnor showed that their images coincide \[4\]. The purpose of this note is to record the following observation which provides a converse to and includes the result of \[5\].

**Theorem.** For any CW complex \( X \), image \( \phi_*(X) \subset \text{image } \psi_*(X) \), image \( \phi_*(X) = \text{image } \psi_*(X) \) if and only if the projective dimension of \( MU_*(X) \otimes Z_{(2)} \) as a \( MU_*(\_ ) \otimes Z_{(2)} \) module is at most one.

Of course, for finite complexes, the analogous statement relating the images of \( \phi^*(X) \) and \( \psi^*(X) \) is also true (but the reader should be warned that the projective dimensions of \( MU_*(X) \otimes Z_{(2)} \) and of \( MU^*(X) \otimes Z_{(2)} \) as \( MU_* \otimes Z_{(2)} \) and \( MU^* \otimes Z_{(2)} \) modules are not in general equal). Since \( \psi^*(X) \) is monic and multiplicative, we obtain the following corollary which may be of interest.

**Corollary.** For finite complexes, there is a natural multiplicative transformation of functors
\[
\psi^*(X)^{-1} \circ \phi^*(X) : MU^*(X) \rightarrow H^*(X; Z) \otimes (N^*)^2.
\]
\( \psi^*(X)^{-1} \circ \phi^*(X) \) is epic if and only if the projective dimension of \( MU^*(X) \otimes Z_{(2)} \) as a \( MU^* \otimes Z_{(2)} \) module is at most one.

**Proof of Theorem.** Given a CW complex \( X \), there is a \(-1\)-connected CW spectrum \( A \) with \( H_*(A; Z) \) free abelian and a stable map \( f : A \rightarrow X \) such that the induced homomorphism in complex bordism (\( f_3 \) in diagram \((*)\))

\[
\begin{array}{ccc}
0 & \rightarrow & H_*(A; Z) \otimes (N_*)^2 \\
& f_1 & \downarrow \psi_*(A) \\
& & H_*(X; Z) \otimes (N_*)^2 \\
(\ast) & & \\
& N_*(A) & \downarrow \psi_*(X) \\
& f_2 & \rightarrow \\
& & N_*(X) \\
& \phi_*(\_ ) & \downarrow \\
& \phi_*(A) & \rightarrow \\
& f_3 & \rightarrow \\
& & MU_*(A) \\
& & \downarrow \phi_*(X) \\
& & MU_*(X) \rightarrow 0
\end{array}
\]
is epic ([2, Proposition 2.4 for the finite case]; [3, Lemma 5 for the generalization]).

From Milnor's result that image $\psi_\ast(\text{point}) = \text{image } \phi_\ast(\text{point})$ and from the fact that $H_\ast(A; Z)$ is free abelian, it is routine to show that the images of $\psi_\ast(A)$ and $\phi_\ast(A)$ coincide. From the commutativity of (1), we have image $\phi_\ast(X) = \phi_\ast(X) \circ f_3 = f_3(\text{image } \phi_\ast(A)) = f_3(\text{image } \psi_\ast(A)) = \text{image } \psi_\ast(X) \circ f_1 \subseteq \text{image } \psi_\ast(X)$. Since $\psi_\ast(X)$ is a monomorphism, image $\phi_\ast(X) = \text{image } \psi_\ast(X) \circ f_1$ is precisely image $\psi_\ast(X)$ if and only if $f_1 = H_\ast(f; Z) \otimes (N_\ast)^2$ is epic. We may identify $Z_2$ as the zero component of the graded $Z_2$ vector space, $(N_\ast)^2$. We have a chain of equivalences:

$$f_1 = H_\ast(f; Z) \otimes (N_\ast)^2 \text{ is epic}$$
$$\Rightarrow H_\ast(f, Z) \otimes Z_2 \text{ is epic}$$
$$\Rightarrow f_3 = H_\ast(f, Z_{(2)}) \cong H_\ast(f; Z) \otimes Z_{(2)} \text{ is epic}$$
$$\Rightarrow \mu(X) \otimes Z_{(2)} \text{ is epic in (**) }$$
$$\Rightarrow \text{the projective dimension of } MU_\ast(X) \otimes Z_{(2)} \text{ as a } MU_\ast \otimes Z_{(2)} \text{ module is at most one.}$$

The first two of these equivalences follow from elementary algebra. The third is by consideration of the commutative diagram (**).

$$MU_\ast(A) \otimes Z_{(2)} \xrightarrow{f_3 \otimes Z_{(2)}} MU_\ast(X) \otimes Z_{(2)} \rightarrow 0$$
$$\downarrow \mu(A) \otimes Z_{(2)} \quad \downarrow \mu(X) \otimes Z_{(2)}$$
$$H_\ast(A; Z_{(2)}) \xrightarrow{f_4} H_\ast(X; Z_{(2)})$$
$$\downarrow 0$$

Since $H_\ast(A; Z)$ is free abelian, the Thom homomorphism $\mu(A)$ is epic. The last equivalence is by Corollary 3.11 $\otimes Z_{(2)}$ of [3] as extended in [2].

ACKNOWLEDGEMENT. The author thanks R. E. Stong for his forbearance. This note grew out of a spurious assertion made to him; it was motivated by A. Dold's review of [5] [Math. Rev. 43 (1972), 746. Abstract #4039].

REFERENCES


Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506