

A GENERALIZATION OF THE DOG BONE SPACE TO E^n

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ABSTRACT. In this paper we construct an upper semicontinuous decomposition of E^n ($n \geq 3$) into points and tame arcs such that the associated decomposition space is distinct from E^n .

The purpose of this paper is to construct an upper semicontinuous decomposition G of E^n , $n \geq 3$, into points and tame arcs such that the decomposition space E^n/G is topologically distinct from E^n . For $n=3$, the example is a modification of R. H. Bing's dog bone space [2]. Unlike the dog bone space, it is not difficult to distinguish our decomposition spaces from Euclidean space.

The main idea for the construction of these decomposition spaces was communicated to us by R. D. Anderson during his visit to the University of Texas in March, 1972. Anderson's idea was roughly the following: Take two disjoint wild Cantor sets in E^n . To each point in one Cantor set correspond a unique point in the other Cantor set and join the two points with an arc that is locally polygonal modulo its end points. The collection of arcs thus obtained can be constructed so that its union is homeomorphic to the product of a Cantor set and an arc. Such arcs for $n \geq 4$ are tame and, hopefully, if the Cantor sets are wild enough and the pairings are chosen cleverly, the resulting upper semicontinuous decomposition of E^n will have a decomposition space distinct from E^n .

The difficulty of proving that such decomposition spaces are not E^n lies in finding a topological property of E^n not shared by the decomposition spaces. We use the following elementary property of E^n .

THEOREM 1. *If C is a Cantor set in E^n ($n \geq 3$), U is an open set containing C , and f and g are maps from a 2-cell D into E^n , then there exist maps f' and g' from D into E^n such that $f'|f^{-1}(E^n - U) = f|f^{-1}(E^n - U)$, $g'|g^{-1}(E^n - U) = g|g^{-1}(E^n - U)$, $f'(f^{-1}(U)) \subset U$, $g'(g^{-1}(U)) \subset U$ and $f'(D) \cap g'(D) \cap C = \emptyset$.*

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PROOF. By adjusting $f|f^{-1}(U)$ and $g|g^{-1}(U)$ if necessary, we may assume that (1) f and g are locally PL modulo $f^{-1}(E^q - U)$ and $g^{-1}(E^q - U)$ respectively, and (2) $f(D) \cap U$ and $g(D) \cap U$ are in general position. For $n \geq 5$, $f(D) \cap g(D) \cap U = \emptyset$ so we may take $f' = f$ and $g' = g$. In the other cases we obtain the required maps f' and g' by composing f and g , respectively, with a space homeomorphism that pushes the tame one ($n=3$) or zero ($n=4$) dimensional set $f(D) \cap g(D) \cap U$ off C .

Wild Cantor sets in E^n . The Cantor sets we use are similar to the generalizations of Antoine's necklace [1] discussed by W. A. Blankinship [4] in 1950.

A wild Cantor set in E^n ($n \geq 3$) may be constructed as the intersection of a nest $M_1 \supset \text{Int } M_1 \supset M_2 \supset \text{Int } M_2 \supset M_3 \cdots$ of PL n -manifolds M_j with boundary such that each component of each M_j is homeomorphic to the solid n -torus $B^2 \times S_1^1 \times S_2^1 \times \cdots \times S_{n-2}^1$ where B^2 is a 2-cell and S_i^1 is a 1-sphere. If A is a component of M_j , then the components A_1, \dots, A_k of M_{j+1} in A are embedded in A in the following canonical way.

Let P_i ($i=1, \dots, n-2$) be the natural projection of the solid n -torus $A \sim B^2 \times S_1^1 \times \cdots \times S_i^1 \times \cdots \times S_{n-2}^1$ onto its factor space $B^2 \times S_i^1$. For each integer i ($i=1, \dots, n-2$) we associate an embedding of k solid n -tori A_1, \dots, A_k in A . Since there exists a homeomorphism of A onto itself which interchanges S^1 factors, the $n-2$ embeddings are actually alike topologically. Let i be fixed. The factor space $B^2 \times S_i^1$ is the large solid

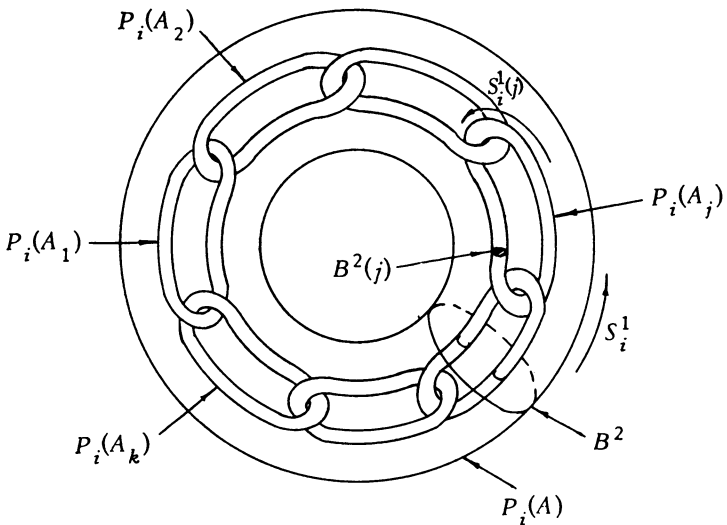


FIGURE 1

3-torus in Figure 1, and the embeddings of the projections $P_i(A_1), \dots, P_i(A_k)$ of the solid n -tori A_1, \dots, A_k in $P_i(A) \sim B^2 \times S_i^1$ are the smaller solid 3-tori linked one to another around the factor S_i^1 as in Figure 1. The solid n -torus A_j is like A in $n-3$ of its factors; in fact, if $B^2(j) \times S^1(j)$ is the solid 3-torus in Figure 1, then $A_j = B^2(j) \times S_1^1 \times \dots \times S_{i-1}^1 \times S_i^1(j) \times S_{i+1}^1 \times \dots \times S_{n-2}^1$. The number k of A_j 's may be arbitrarily large but always at least 3. The embedding of the solid n -tori $\{A_j\}$ in the solid n -torus A will be referred to as an *Antoine embedding*.

We start the construction of a wild Cantor set by taking M_1 to be any solid n -torus in E^n . We would like to make the diameter of the components of M_2 in M_1 small relative to the diameter of M_1 by using an Antoine embedding of solid n -tori in $M_1 = A$. However, no matter how large the integer k we can make a component A_j of M_2 small only in two of its coordinates, namely $B^2(j)$ and $S^1(j)$ of the previous paragraph. Thus we may make all the components small at the second stage only if $n=3$. The remedy for the difficulty for $n>3$ is to use $n-2$ stages to make components small relative to the diameter of M_1 . That is, link the solid n -tori of M_2 about the first S^1 factor of M_1 (use P_1 of the previous paragraph), then in each component of M_2 link the solid n -tori about the second S^1 factor, etc. Thus the components of M_{n-1} may be made of arbitrarily small diameter, and by repeating the process we can insure that $\cap M_i$ is a Cantor set.

To show that the Cantor set $M = \cap M_i$ is wild, Blankinship [4] computed explicitly $\pi_1(E^n - M)$ and showed that the group was nontrivial by representing it nontrivially in the symmetric group S_6 . We are more interested in geometry than its amusing derivatives; consequently, we proceed alternatively by using the following definition and lemma.

DEFINITION 1. Let A be a solid n -torus $B^2 \times S_1^1 \times \dots \times S_{n-2}^1$ in E^n . A PL map f from a 2-cell D into E^n is *regular with respect to A* if

- (1.1) $f(D)$ and $\text{Bd } A$ are in general position,
- (1.2) $f|_{\text{Bd } D}$ is a nontrivial loop in $\text{Bd } A$,
- (1.3) there exists a collar C of $\text{Bd } D$ in D such that $f(\text{Int } C) \subset \text{Int } A$, and
- (1.4) if K is a component of $f^{-1}(f(\text{Int } D) \cap \text{Bd } A)$ (K is necessarily a simple closed curve by (1.1)) then $f|_K$ is a trivial loop in $\text{Bd } A$.

LEMMA 1. If A is a solid n -torus ($n \geq 3$) $B^2 \times S_1^1 \times \dots \times S_{n-2}^1$ in E^n , the solid n -tori A_1, \dots, A_k are Antoine embedded in A , f is a map from a 2-cell D into E^n such that f is regular with respect to A and $f(D)$ and $\cup \text{Bd } A_i$ are in general position, then there exist an integer j and a subdisk D^* of D such that $f|_{D^*}$ is regular with respect to A_j .

PROOF. Since A_1, \dots, A_k are Antoine embedded in A , for some i the natural projection map P_i of $A \sim B^2 \times S_1^1 \times \dots \times S_i^1 \times \dots \times S_{n-2}^1$ onto

its factor space $B^2 \times S_i^1$ takes the A_j 's to the solid 3-tori linked around S_i^1 as indicated in Figure 1. We first show that there exists a component K of $f^{-1}(\cup \text{Bd } A_j)$ and an integer j such that $f|_K$ is nontrivial in $\text{Bd } A_j$. We assume the contrary and let a_0, a_1, \dots, a_{n-2} be a free basis for the free abelian group $\pi_1(\text{Bd } A)$. We may take $a_0 = [\text{Bd } B^2]$, $a_1 = [S_1^1], \dots, a_{n-2} = [S_{n-2}^1]$ so that a_1, \dots, a_{n-2} is a free basis for the free abelian group $\pi_1(A)$. By (1.3), (1.4) and our supposition, we may adjust f by cutting off trivial loops on $(\cup \text{Bd } A_j) \cup \text{Bd } A$ to obtain a map f' from D into A such that $f'|_{\text{Bd } D} = f|_{\text{Bd } D}$ and $f'(D) \cap (\cup A_j) = \emptyset$. Thus $f|_{\text{Bd } D}$ is a nontrivial loop in $\text{Bd } A$ (1.2) and trivial in A and, consequently, has the reduced form a_0^r for some nonzero integer r . It follows that $P_i f'|_{\text{Bd } D}$ is nontrivial in the boundary of the solid 3-torus $B^2 \times S_i^1$ and $P_i f'(D)$ misses each of the solid 3-tori $P_i(A_j)$. These facts contradict that the $P_i(A_j)$'s link one to another around the S_i^1 factor of $B^2 \times S_i^1$ (see [3, Theorems 9 and 11] for the techniques to prove this 3-space result). Thus there are components K of $f^{-1}(\cup \text{Bd } A_j)$ such that $f|_K$ is nontrivial in $\cup \text{Bd } A_j$.

Let K be an innermost such component on D , let D^* be the subdisk in D that K bounds and let j be the integer such that $f(K) \subset \text{Bd } A_j$. We complete the proof by showing that $f|_{D^*}$ is regular with respect to A_j . Note that it is only necessary to show that there exists a collar C^* of D^* such that $f(\text{Int } C^*) \subset \text{Int } A_j$. We suppose the contrary; that is, suppose there exists a collar C^* of D^* such that $f(\text{Int } C^*) \subset (\text{Ext } A_j) \cap (\text{Int } A)$. Since each component $J \neq \text{Bd } D^*$ of $(f|_{D^*})^{-1}((\cup \text{Bd } A_m) \cup \text{Bd } A)$ has the property that $f|_J$ is trivial on $(\cup \text{Bd } A_m) \cup \text{Bd } A$, we may adjust the map $f|_{D^*}$ to obtain a map f' such that $f'(\text{Int } D^*) \subset (\text{Int } A) - \cup A_m$ and $f'|_{\text{Bd } D^*} = f|_{\text{Bd } D}$. The loop $f|_{\text{Bd } D}$ is thus trivial in A and, consequently, cannot contain the factors $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n-1}$ in its reduced form. Thus the reduced form of $f|_{\text{Bd } D^*}$ contains only the factors $a_0(j) = [\text{Bd } B^2(j)]$ and $a_i(j) = [S_i^1(j)]$ in $\pi_1(\text{Bd } A_j)$ (see Figure 1), and since $f|_{\text{Bd } D^*}$ is nontrivial on $\text{Bd } A_j$, the reduced form of $f|_{\text{Bd } D^*}$ contains at least one of $a_0(j)$ and $a_i(j)$ as factors. It follows that $P_i f'|_{\text{Bd } D^*}$ is nontrivial on $\text{Bd } P_i(A_j)$ and $P_i f'(\text{Int } D^*) \cap (\cup_m P_i(A_m)) = \emptyset$. Since $P_i f'(\text{Int } D^*) \cap P_i(A_j) = \emptyset$ we have that $P_i f'|_{\text{Bd } D^*}$ contains the factor $a_i(j)$, but this is impossible since some $P_i(A_m)$ links $P_i(A_j)$ denying that $f'(\text{Int } D^*) \cap P_i(A_m)$ is empty. Hence D^* contains a collar C^* such that $f(\text{Int } C^*) \subset \text{Int } A_j$ and $f|_{D^*}$ is regular with respect to A_j .

We will use Lemma 1 and Theorem 1 to establish that our decomposition spaces must fail to be E^n . We first use Lemma 1 to show that the Cantor set $\cap M_j$ is wild in E^n .

Suppose $\cap M_j$ is not wild in E^n ; then with a slight adjustment of B^2 in M_1 we may assume that B^2 misses $\cap M_j$. We realize the adjusted B^2 as

the image of a disk D under a PL map f . We may assume that $f(D)$ is in general position with respect to each $\text{Bd } M_j$. Lemma 1 is used repeatedly to obtain a sequence $D = D_1 \supset D_2 \supset D_3 \supset \dots$ of subdisks of D and a sequence $A_{i_1} \supset A_{i_1 i_2} \supset A_{i_1 i_2 i_3} \supset \dots$ of solid n -tori such that $f|_{D_j}$ is regular with respect to the solid n -torus $A_{i_1 \dots i_j}$, where $A_{i_1 \dots i_j}$ is a component of M_j . It follows that $f(\bigcap_j D_j) = \bigcap A_{i_1 \dots i_j}$, which is a point of the Cantor set $\bigcap M_j$. Thus a contradiction is reached and we have that $\bigcap M_j$ must be wild.

The construction. The Cantor sets we use are very much like the Antoine-Blankinship type. We realize each Cantor set as the intersection of a nest $M_1 \supset \text{Int } M_1 \supset M_2 \supset \text{Int } M_2 \supset \dots$ of PL n -manifolds with boundary in E^n such that each component of each M_j is a solid n -torus $B^2 \times S^1_1 \times \dots \times S^1_{n-2}$. If $A \sim B^2 \times S^1_1 \times \dots \times S^1_{n-2}$ is a component of M_j then there are k^2 components A_{11}, \dots, A_{kk} of M_{j+1} embedded in A so that for some i the natural projection map P_i of A onto its factor space $B^2 \times S^1_i$ carries the solid n -tori A_{11}, \dots, A_{kk} onto the solid 3-tori $P_i(A_{11}), \dots, P_i(A_{kk})$ embedded in $P_i(A) \sim B^2 \times S^1_i$ as in Figure 2. As Figure 2 illustrates, for each j the solid n -tori A_{j1}, \dots, A_{jk} are very close approximations to each other; in fact, if for each j we choose $i_j \in \{1, \dots, k\}$ then $A_{1i_1}, \dots, A_{ki_k}$ are Antoine embedded in A . The components of M_k are made small as $k \rightarrow \infty$ in the same way as for the Antoine-Blankinship Cantor sets; that is, the S^1 factors of the solid n -torus A that the A_{ij} link around alternate as $k \rightarrow \infty$.

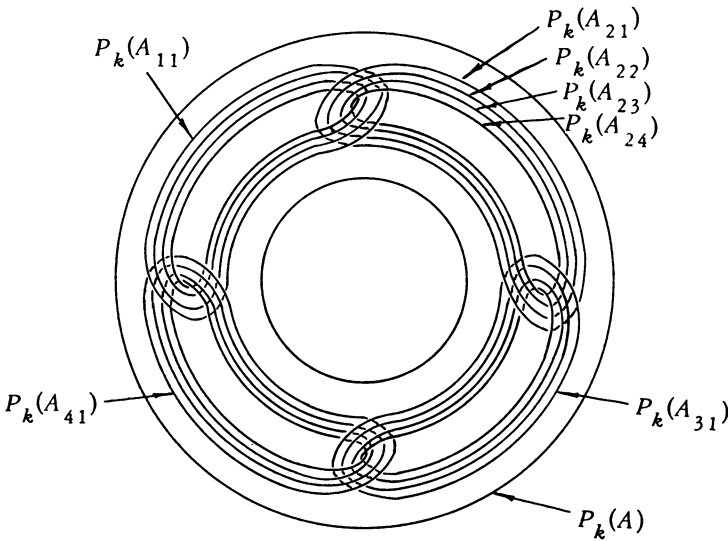


FIGURE 2

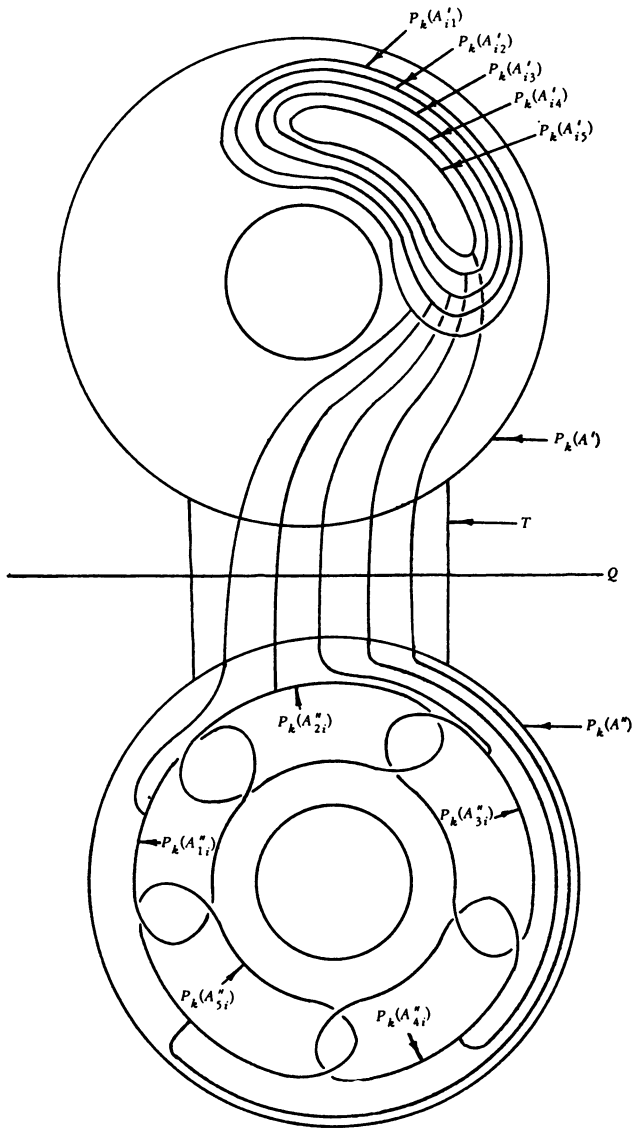


FIGURE 3

For convenience we locate our disjoint Cantor sets on opposite sides of an $(n-1)$ -dimensional hyperplane Q in E^n . We denote the manifolds used to describe the Cantor set above Q with primed letters and the manifolds used to describe the Cantor set below Q by double primed letters. The upper semicontinuous decomposition G of E^n into points and tame arcs is now easy to describe.

The n -manifolds M'_1 and M''_1 are solid n -tori; that is, M'_1 and M''_1 each have one component homeomorphic to $B^2 \times S^1_1 \times \cdots \times S^1_{n-2}$. We connect M'_1 and M''_1 with an n -tube, T_1 , and we denote the union of M'_1 , M''_1 , and the tube T_1 by M_1 . Each component A'_{ij} of M'_2 is connected to a unique component A''_{ji} of M''_2 by an n -tube T_{ij} in $\text{Int } M_1$ that runs straight through the tube T_1 (see Figure 3). We denote the union of A'_{ij} , A''_{ji} , and T_{ij} by A_{ij} . The T_{ij} 's are constructed disjointly and the union of the k^2 n -manifolds A_{ij} is M_2 . Inductively, if $A = A' \cup A'' \cup T$ is a component of M_k where A' , A'' are solid n -tori and T is an n -tube connecting them, then the components, $\{A_{ij}\}$, of M_{k+1} in A are obtained by connecting each of the k^2 solid n -tori A'_{ij} in A' to a unique solid n -torus A''_{ji} in A'' by an n -tube T_{ij} in $\text{Int } A$ that runs straight through the n -tube T . The T_{ij} 's are constructed disjointly as thin regular neighborhoods of PL arcs. Figure 3 illustrates the iterated step in which the n -tubes connecting the A'_{ij} 's and A''_{ji} 's are schematically pictured as connecting $P_k(A'_{ij})$'s and $P_k(A''_{ji})$'s.

The nondegenerate elements of our upper semicontinuous decomposition G of E^n are the components of $\cap M_i$. Since the components of M'_i and M''_i become small as i gets large and, in the iterative step, we require that each n -tube T_{ij} be thin and run straight through the n -tube T of the preceding stage, each nondegenerate element of G is an arc that is locally polygonal modulo its end points. It is well known and an easy exercise to show that such arcs are tame if $n \geq 4$. For $n=3$, we can insure that the arcs are tame by also requiring that the 3-tubes do not pass through the holes of the solid 3-tori they connect. Thus each arc will be unknotted at its ends.

The following lemma will be used iteratively to help establish that E^n/G is not topologically E^n .

LEMMA 2. *If $A = A' \cup A'' \cup T$ is a component of M_k from the description of G above, where A' and A'' are solid n -tori and T is an n -tube connecting them, and f and g are PL maps from a 2-cell D into E^n such that $f(D) \cup g(D)$ and $(\cup \text{Bd } A'_{ij}) \cup (\cup \text{Bd } A''_{ji})$ are in general position, f is regular with respect to A' , and g is regular with respect to A'' , then there exist subdisks D_0 and E_0 of D and a component $A_{ij} = A'_{ij} \cup T_{ij} \cup A''_{ji}$ of M_{k+1} in A such that $f|D_0$ is regular with respect to A'_{ij} and $g|E_0$ is regular with respect to A''_{ji} .*

PROOF. There exists an integer i such that for all m there exists a subdisk D_m of D such that $f|_{D_m}$ is regular with respect to A'_{im} . For suppose the contrary, then for each i we may pick an integer j_i such that $f|_{D'}$ is not regular with respect to A'_{ij_i} for each subdisk D' of D . But $A'_{1j_1}, \dots, A'_{kj_k}$ form an Antoine embedding in A' , thus we have a contradiction to Lemma 1.

The solid n -tori $A''_{1i}, \dots, A''_{ki}$ form an Antoine embedding in A'' . By Lemma 1, there exist a subdisk E_0 of D and an integer j such that $g|_{E_0}$ is regular with respect to A''_{ji} . The proof is completed by setting $D_0 = E_0$.

THEOREM 2. *If G is the upper semicontinuous decomposition described above then E^n/G is not topologically E^n ($n \geq 3$).*

PROOF. We suppose that $E^n/G \sim E^n$ and let P be the natural projection map of E^n onto $E^n/G \sim E^n$ associated with the decomposition G . Let f and g be homeomorphisms from a disk D onto the B^2 factors of the solid n -tori M'_1 and M''_1 , respectively. We may assume that $f(D)$ and $g(D)$ miss the n -tube T_1 and that f and g are regular with respect to M'_1 and M''_1 , respectively.

We now use our supposition and Theorem 1 to adjust f and g to obtain maps f'', g'' and an integer such that $f''(D)$ and $g''(D)$ do not intersect a common component of M_j and f'' and g'' are regular with respect to M'_1 and M''_1 , respectively. By Theorem 1, $Pf(D)$ and $Pg(D)$ may be adjusted in $P(\text{Int } M_1)$ to obtain maps f' and g' from D into $P(\text{Int } M_1)$ such that $f'|_{\text{Bd } D} = Pf|_{\text{Bd } D}$, $g'|_{\text{Bd } D} = Pg|_{\text{Bd } D}$, and $f'(D) \cap g'(D) \cap P(\cap M_i) = \emptyset$. Since $f'(D) \cap g'(D) \cap P(\cap M_i) = \emptyset$, there exists an integer j such that $f'(D)$ and $g'(D)$ do not intersect a common component of $P(M_j)$. Since the components of $\cap M_i$ are cellular, there exists a collection of n -cells $\{C_i\}$ such that $C_i \subset \text{Int } M_j$ and each component of $\cap M_j$ lies in the interior of some C_i . The collection $\{\text{Int } P(C_i)\}$ covers $P(\cap M_i)$, thus there exists an integer k such that each component of $P(M_{j+k})$ lies in $\text{Int } P(C_i)$ for some i . Let F and G be, respectively, the component of $(f')^{-1}(E^n - P(M_{j+k}))$ and the component of $(g')^{-1}(E^n - P(M_{j+k}))$ that contain $\text{Bd } D$. Since $\cap M_i \subset \text{Int } M_{j+k}$, P^{-1} is a well defined map on $f'(F) \cup g'(G)$. We obtain the map f'' by extending $P^{-1}f'|_F$ to D so that $f''(K) \subset \text{Int } C_i$ where K is a component of $D - F$ and C_i is an n -cell that contains $P^{-1}f'(\text{Bd } K)$. The map g'' is obtained similarly.

The existence of the integer j and the maps f'' and g'' is contrary to Lemma 2, however. By applying Lemma 2 repeatedly, we obtain nests $D = F_1 \supset F_2 \supset \dots \supset F_j$, $D = G_1 \supset G_2 \supset \dots \supset G_j$ of subdisks of D , and nests $A_1 \supset A_2 \supset A_3 \supset \dots \supset A_j$ such that A_i is a component of M_i and

$$f''(F_i) \cap A_i \neq \emptyset \text{ and } g''(G_i) \cap A_i \neq \emptyset \quad \text{for } i = 1, \dots, j.$$

Thus $f''(D)$ and $g''(D)$ intersect a common component of M_j .

Assuming that $E^n/G \sim E^n$ leads to a contradiction, thus E^n/G is not topologically E^n .

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