PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 2, July 1973

ASYMPTOTIC INVERSION OF LAPLACE TRANSFORMS: A CLASS OF COUNTEREXAMPLES

JOHN S. LEW

ABSTRACT. Let f be a complex-valued locally integrable function on $[0, +\infty)$, and let Lf be its Laplace transform, whenever and wherever it exists. We review some known methods, exact and approximate, for recovering f from Lf. Since numerical algorithms need auxiliary information about f near $+\infty$, we note that the behavior of f near $+\infty$ depends on the behavior of Lf near $0+$, hence that our ability to retrieve f is limited by the class of *momentless* functions, namely, all functions f such that $Lf(s)$ converges absolutely for $\mathrm{Re}(s)>0$ and satisfies

$$Lf(s) = o(s^n) \quad \text{near } 0+ \qquad \text{for } n = 0, 1, 2, \cdots.$$

We discuss the space Z of momentless functions and complex distributions, then construct a family of elements in this space which defy various plausible conjectures.

1. **Introduction.** Let f be a complex-valued locally integrable function on $[0, +\infty)$, and let

$$L[f; s] = \int_0^\infty \exp(-st)f(t)\, dt \tag{1.1}$$

be its Laplace transform, whenever and wherever this integral exists. Indeed [**21**, pp. 96–102] for some $\sigma_a(f)$ and $\sigma_c(f)$ with $-\infty \leqq \sigma_c(f) \leqq \sigma_a(f) \leqq +\infty$ there are maximal half-planes $\mathrm{Re}(s)>\sigma_a(f)$ and $\mathrm{Re}(s)>\sigma_c(f)$ in which respectively $L[f; s]$ is absolutely convergent and conditionally convergent. If these half-planes are nonvoid, then $L[f, s]$ is also holomorphic at least in $\mathrm{Re}(s)>\sigma_c(f)$, and $f(t)$ is uniquely determined by $L[f; s]$ except on a set of measure zero [**21**, pp. 99, 108].

An important step in many problems is the inversion of a Laplace transform, that is, the recovery of $f(t)$ from $L[f; s]$. Sometimes this can be accomplished exactly through transform tables (e.g. [**5**]), inversion formulas ([**3**, p. 286], [**6**], [**21**, pp. 108, 141]), or convergent series ([**3**, pp. 301–305], [**17**, p. 97], [**18**], [**19**, Chapter 9]). However many inversions

Received by the editors September 22, 1972.

AMS (MOS) subject classifications (1970). Primary 41A60, 44A10; Secondary 30A84, 44A40.

Key words and phrases. Laplace transform, asymptotic inversion, inverse Abelian theorem, Tauberian theorem, Mellin series.

employ numerical techniques ([**1**], [**2**]), which typically lose accuracy for large t; hence such algorithms require further information which describes $f(t)$ approximately near $+\infty$. Theorems which derive the limiting behavior of $f(t)$ from that of $L[f; s]$ are called respectively *Tauberian* or *inverse Abelian* according as they involve extra hypotheses on $f(t)$ or on $L[f; s]$.

If $f(t)$ can be expressed by the inversion integral [**21**, p. 108]

$$f(t) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \exp(ts)L[f; s]\, ds, \tag{1.2}$$

if $L[f; s]$ can be continued analytically to the left, and if the contour of (1.2) can be moved sufficiently in that direction, then the behavior of $f(t)$ near $+\infty$ is determined by that of $L[f; s]$ near its rightmost singularities. If these singular points are all poles then their contributions are simple residues and [**4**, p. 110]

$$f(t) \sim \sum_{m=0}^{\infty} P_m(t)\exp[a(m)t] \quad \text{near } +\infty, \tag{1.3}$$

with P_m a polynomial for each m, and $\mathrm{Re}[a(m)]\downarrow -\infty$ as $m \to \infty$. An essential singularity of $L[f; s]$ yields a Taylor expansion for P_m [**3**, p. 488].

If $L[f; s]$ has a branch point s_0 as its unique rightmost singularity, then s_0 may be shifted to the origin without loss of generality. Thus for any positive a we can infer $\int_0^t f(u)\, du \sim ct^a/\Gamma(1+a)$ near $+\infty$ by the Tauberian theorem of Karamata [**19**, p. 197], given that $L[f; s] \sim cs^{-a}$ near $0+$ and that $f(t)+kt^{a-1} \geqq 0$ for some k. Also for any complex a we can infer that $f(t) \sim ct^{a-1}/\Gamma(a)$ near $+\infty$ by inverse Abelian theorems of Doetsch, given that $L[f; s] \sim cs^{-a}$ near 0 in a sector $|\arg s| \leqq \theta$, where either $\theta > \pi/2$ or $\theta = \pi/2$, and in the latter case $L[f; s]$ satisfies further conditions on the imaginary axis. For generalized functions some results of this kind have been proved by Lavoine [**12**], involving the regularized functions ct^{a-1}; for $\log f(t)$ some estimates near $+\infty$ have been obtained by Wagner ([**17**], [**18**], [**25**]), describing still more singular behavior.

To get sharper results at a branch point, we consider series

$$f(t) \sim \sum_{m=0}^{\infty} P_m(\log t)t^{a(m)} \quad \text{near } +\infty \tag{1.4}$$

with P_m and $\mathrm{Re}[a(m)]$ as in (1.3). Expansions near $+\infty$ of this form, or near $0+$ with $\mathrm{Re}[a(m)]\uparrow +\infty$, were originally treated, it seems, by Mellin [**13**], and are thus called *Mellin series* or *expansions* by the author. If the function f has a Mellin series near $+\infty$ then this series for f, and certain values of its Mellin transform, determine systematically a Mellin series for $L[f; s]$ near $0+$ [7]; while this series for $L[f; s]$ and the *existence* of an expansion (1.4) determine uniquely the Mellin series for $f(t)$ near $+\infty$ ([**8**], [**9**]). The existence and form of a series (1.4) follows by two inverse Abelian theorems of Doetsch ([**4**, pp. 150–160], [**9**], [**11**]) from assumptions on $L[f; s]$ in a sector $|\arg s| \leqq \theta$, where either $\theta > \pi/2$ or $\theta = \pi/2$, and in the latter case $L[f; s]$ satisfies further conditions on the imaginary axis. However these assumptions on $L[f, s]$ are not necessary [**9**, Example 4]. These results of Doetsch have also been extended in work of Riekstiņa ([**23**], [**24**]).

To explore all possibilities for theorems of this kind, we remark that if g is rapidly decreasing near $+\infty$ then $L[g; s]$ can be expanded by moments:

$$(1.5)\quad L[g; s] \sim \sum_{n=0}^{\infty} \mu_n(-s)^n/n! \quad \text{near } 0+ \qquad \text{with} \quad \mu_n = \int_0^\infty t^n g(t)\, dt.$$

Thus any transformable g will be called a *momentless* function if $\sigma_a(g) \leqq 0$ and

$$(1.6)\qquad L[g; s] = o(s^n) \quad \text{near } 0+ \qquad \text{for all } n = 0, 1, 2, \cdots.$$

A nontrivial example from standard tables [**5**, p. 158] is

$$(1.7)\qquad \begin{aligned} g(t) &= t^{-1/2}\cos(kt)^{1/2} \quad \text{with } k > 0, \\ L[g; s] &= (\pi/s)^{1/2}\exp(-k/4s). \end{aligned}$$

If g is a function of this kind then $L[f; s]$ and $L[f+g; s]$ have identical Mellin series near $0+$. Thus the Mellin series for f is not recoverable unless all permissible g are rapidly decreasing under the set of hypotheses for a conjectured theorem. We shall therefore construct a class of momentless functions and distributions through which we may eliminate a number of conjectures on asymptotic inversion.

2. **Notation.** We shall construct the desired counterexamples on $[0, +\infty)$ as a family of functions and measures, but can introduce the associated concepts more easily in a space of generalized functions. Indeed if D'_+ is the space of Schwartz distributions on $(-\infty, +\infty)$ with support in $[0, +\infty)$ then D'_+ is a commutative algebra over the complex field under "pointwise" addition, scalar multiplication, and the standard convolution ([**15**, pp. 113, 121], [**22**, pp. 122–130]). This convolution $f * g$ for elements of D'_+ extends the definition

$$(2.1)\qquad [f * g](t) = \int_0^t f(t-u)g(u)\, du$$

for functions on $[0, +\infty)$ [**15**, p. 115].

Within D'_+ let e represent the Dirac delta "function", so that e is the identity for this algebra, and let 1_+ denote the Heaviside step function, so that

$$[1_+ * f](t) = \int_0^t f(u)\, du \tag{2.2}$$

for functions on $[0, +\infty)$. Then we can define

$$f^{*0} = e, \qquad f^{*1} = f, \qquad f^{*n+1} = f * f^{*n} \tag{2.3}$$

for any f in D'_+ and all $n = 1, 2, \cdots$. Moreover D'_+ is closed under

$$f \to 1_+ * f, \qquad f \to df/dt, \tag{2.4}$$

and the first of these mappings is the inverse of the second.

For any element f of D'_+ and any complex $s = c + iu$ the Laplace transform $L[f; s]$ is defined ([**15**, p. 217], [**22**, p. 222]) as the Fourier transform

$$\int_0^\infty \exp(-itu)\exp(-ct) f(t)\, dt \tag{2.5}$$

whenever $\exp(-ct)f(t)$ is in the space S', so that (2.5) is a well-defined entity. Then for some value $\sigma(f)$, either real or $\pm\infty$, the transform $L[f; s]$ is defined and analytic on $\mathrm{Re}(s) > \sigma(f)$, and for functions on $[0, +\infty)$ this half plane of existence includes the preceding $\mathrm{Re}(s) > \sigma_a(f)$ ([**15**, p. 218], [**22**, p. 223]).

Now consider the set A of all f in D'_+ such that $L[f; s]$ is defined in this sense for $\mathrm{Re}(s) > 0$ at least and such that

$$L[f; s] = O(s^k) \quad \text{for some real } k \tag{2.6}$$

as $s \to 0$ in this half plane. Clearly A is a subalgebra of D'_+ by the convolution theorem ([**15**, p. 222], [**22**, p. 240]), and is closed under the mappings (2.4) by the identities ([**15**, pp. 222–223], [**22**, p. 228])

$$L[1_+ * f; s] = s^{-1}L[f; s], \qquad L[df/dt; s] = sL[f; s]. \tag{2.7}$$

Call f *momentless* if it lies in A and satisfies (1.6); define Z as the set of all such f. Then Z is an ideal in A by (1.6) and (2.6), while Z is closed under (2.4) by (1.6) and (2.7).

Within A denote by J the space of all elements f which correspond to locally integrable functions, modulo the space of all functions which vanish except on null sets. The introduction of J offers a criterion for A: if f is given in D'_+ then f is also in A whenever $1_+^{*n} * f$ is in J for some $n = 0, 1, 2, \cdots$, and

$$[1_+^{*n} * f](t) = o(t^k) \quad \text{near } +\infty \tag{2.8}$$

for some $k>0$. Indeed, under these conditions $L[1_+^{*n} * f; s]$ is absolutely convergent on $\mathrm{Re}(s)>0$ and is $o(s^{-k-1})$ as $s\to 0$ ([**19**, p. 182], [**22**, p. 249]); so that $L[f; s]$ satisfies (2.6) by use of (2.7). However this criterion is not necessary, for

$$g(t) = \sum_{n=0}^{\infty} (d/dt)^n e(t-n) \tag{2.9}$$

is in A, but no $1_+^{*n} * g$ is in J.

Let M be the set of all f in A which correspond to complex measures on $[0, +\infty)$, namely, those for which $1_+ * f$ has locally bounded variation. Then M is a subalgebra of A [**19**, p. 84]; measure algebras are discussed in standard works ([**10**, pp. 141–150], [**14**, pp. 13–17]). Let C^n be the set of all f in J which have n continuous derivatives on $(-\infty, +\infty)$, for all $n=0, 1, 2, \cdots$ or ∞. Then J is an ideal in M [**10**, p. 143], all C^n are ideals in M [**15**, p. 122], and C^∞ is closed under (2.4).

Finally, we collect these remarks on algebraic structure to obtain the following ideals in the system M:

$$Z \cap M, Z \cap J, \qquad Z \cap C^n \quad \text{for } n=0, 1, \cdots, \infty. \tag{2.10}$$

Therefore we can generate elements of $Z\cap M$ with arbitrary preassigned smoothness from a special family $\{h_{a,x}\}$ with a and x suitable real numbers. Also we can construct more singular elements of Z by repeated differentiation of $h_{a,x}$.

3. **Construction.** For any real a and x the expression

$$K(a, x, z) = (1-z)^{-1-a} \exp[xz/(z-1)] \tag{3.1}$$

is analytic in the complex z plane cut from 1 to $+\infty$, and is the generating function [**16**, equation (5.1.9)] for the generalized Laguerre polynomials $L_n^{(a)}(x)$, so that

$$K(a, x, z) = \sum_{n=0}^{\infty} L_n^{(a)}(x) z^n \quad \text{for} \quad |z| < 1. \tag{3.2}$$

Moreover, by Fejer's formula [**16**, equation (8.22.1)],

$$\begin{aligned} L_n^{(a)}(x) = \pi^{-1} \exp(x/2) x^{(-2a-1)/4} \cdot n^{(2a-1)/4} \cdot \cos[2(nx)^{1/2} - (2a+1)\pi/4] \\ + O[n^{(2a-3)/4}] \quad \text{as } n \to +\infty \end{aligned} \tag{3.3}$$

uniformly on any compact interval in $0<x<+\infty$.

On $|z|\leqq 1$ the series (3.2) converges absolutely for $a<-\frac{3}{2}$ by the last formula, and conditionally for $a=-\frac{3}{2}$ by Littlewood's theorem [**21**, p. 215]. However for each positive x the set

$$\{2(nx)^{1/2} - (2a+1)\pi/4 : n = 0, 1, 2, \cdots\} \quad \text{modulo } 2\pi \tag{3.4}$$

is dense in $[0, 2\pi)$, so that, with any positive δ and n_0,

$$|L_n^{(a)}(x)| \geqq (1-\delta)\pi^{-1}\exp(x/2)\cdot x^{(-2a-1)/4}\cdot n^{(2a-1)/4} \tag{3.5}$$

for some $n>n_0$. Thus $L_n^{(a)}(x)$ for large n is not $O(n^r)$ unless $r\geqq(2a-1)/4$.

For any fixed real a and positive x, letting $e(t)$ be the Dirac delta function, we construct

$$h_{a,x}(t) = \sum_{n=0}^{\infty} L_n^{(a)}(x)e(t-n). \tag{3.6}$$

By Abel's theorem [**21**, pp. 27–28] if $a\leqq-\frac{3}{2}$ then

$$[1_+ * h_{a,x}](+\infty) = \sum_{n=0}^{\infty} L_n^{(a)}(x) = \lim_{z\to 1-} K(a, x, z) = 0. \tag{3.7}$$

By this relation and (3.3), if a is arbitrary then

$$[1_+ * h_{a,x}](t) = O[t^{(2a+3)/4}] \quad \text{as } t\to+\infty. \tag{3.8}$$

Hence by (2.8) these $h_{a,x}$ are elements of M with support on the nonnegative integers. Moreover these $h_{a,x}$ are elements of Z, since if $\mathrm{Re}(s)>0$ then

$$\begin{aligned} L[h_{a,x}; s] &= \sum_{n=0}^{\infty} L_n^{(a)}(x)\exp(-ns) = K(a, x, \exp(-s)) \\ &= o(s^m) \quad \text{near } 0+ \qquad \text{for } m = 0, 1, 2, \cdots. \end{aligned} \tag{3.9}$$

EXAMPLE 1. Before finding this construction the author advanced the conjecture that if f were an element of Z which was bounded as a measure on $[0, +\infty)$ then

$$[1_+ * f](+\infty) - [1_+ * f](t) = o(t^{-n}) \quad \text{near } +\infty \qquad \text{for all } n>0. \tag{3.10}$$

However if $a<-\frac{3}{2}$ then $h_{a,x}$ is a bounded measure on $[0, +\infty)$ by (3.3), and $h_{a,x}$ is algebraically decaying near $+\infty$ by (3.5). Indeed some multiple of $h_{a,x}$, by (3.7), is the difference of two probability measures on the integers; hence these measures differ by a momentless distribution which decays no faster than $t^{(2a-1)/4}$.

EXAMPLE 2. One might suppose that matters would improve for functions f in $Z\cap L^1(-\infty, +\infty)$. However let $f=g * h_{a,x}$, where g is intuitively any function in $L^1[0, +\infty)$, or technically any function in $J\cap L^1(-\infty, +\infty)$. Then by (2.10), f is in $Z\cap J$ for all real a, and by (3.3), f is in $Z\cap L^1(-\infty, +\infty)$ for $a<-\frac{3}{2}$. Moreover if g is unbounded at the origin and is bounded outside each neighborhood of zero, then f or some multiple is unbounded at $t=0, 1, 2, \cdots$ and is the difference between two probability densities.

EXAMPLE 3. One might expect more from the intersection of Z with some Sobolev space. However if g is C^∞ with support in $(0, 1)$ and if

$f=g*h_{a,x}$ with $h_{a,x}$ as defined, then f is in $Z\cap C^\infty$ by (2.10) and f is in $L^p(-\infty, +\infty)$ by (3.3) whenever

$$\sum_{n=1}^{\infty} n^{(2a-1)p/4} < +\infty, \tag{3.11}$$

hence for all p in $[1, +\infty]$ whenever $a<-\frac{3}{2}$. Moreover $f^{(m)}=g^{(m)} * h_{a,x}$, so that $f^{(m)}$ has the same properties as f, and thus $f^{(m)}$ is in $L^p(-\infty, +\infty)$ for all p in $[1, +\infty]$ and all $m=0, 1, 2, \cdots$. At the same time f and its derivatives decay no faster than $t^{(2a-1)/4}$.

These examples show that conditions on f of smoothness and integrability cannot produce estimates of $f(t)$ near $+\infty$ from estimates of $L[f;s]$ near $0+$. Indeed for any fixed a and x, the numbers

$$\{L_n^{(a)}(x)\colon n = 0, 1, 2, \cdots\} \tag{3.12}$$

change sign according to (3.3), so that the functions $f=g*h_{a,x}$ oscillate systematically as $t\to+\infty$. Thus the positivity condition of Karamata's theorem serves to exclude some elements of Z. However this theorem applies only to functions f for which $1_+ * f$ grows algebraically near $+\infty$, whence new results might well arise from other hypotheses under which f oscillates negligibly near $+\infty$.

References

1. R. E. Bellman, R. E. Kalaba and J. A. Lockett, *Numerical inversion of the Laplace transform: Applications to biology, economics, engineering and physics*, American Elsevier, New York, 1966. MR **34** #5282.

2. J. W. Cooley, P. A. W. Lewis and P. D. Welch, *The fast Fourier transform algorithm: programming considerations in the calculation of sine, cosine, and Laplace transforms*, J. Sound Vib. **12** (1970), 315–337.

3. G. Doetsch, *Handbuch der Laplace-Transformation*. Band I: *Theorie der Laplace-Transformation*, Verlag Birkhäuser, Basel, 1950. MR **13**, 230.

4. ———, *Handbuch der Laplace-Transformation*. Band II: *Andwendungen der Laplace-Transformation*, Verlag Birkhäuser, Basel, 1955. MR **18**, 35.

5. A. Erdélyi et al., *Tables of integral transforms*. Vol. I, McGraw-Hill, New York, 1954. MR **15**, 868.

6. H. Goldenberg, *The evaluation of inverse Laplace transforms without the aid of contour integration*, SIAM Rev. **4** (1962) 94–104. MR **25** #397.

7. R. A. Handelsman and J. S. Lew, *Asymptotic expansion of Laplace transforms near the origin*, SIAM J. Math. Anal. **1** (1970), 118–130. MR **41** #4142.

8. ———, *Asymptotic expansion of Laplace convolutions for large argument*, SIAM Rev. **13** (1971), 269.

9. ———, *Asymptotic expansion of Laplace convolutions for large argument and tail densities for certain sums of random variables*, SIAM J. Math. Anal. (to appear).

10. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, R.I., 1957. MR **19**, 664.

11. T. E. Hull and C. Froese, *Asymptotic behavior of the inverse of a Laplace transform*, Canad. J. Math. **7** (1955), 116–125. MR **16,** 584.

12. J. Lavoine, *Sur les théorèmes abéliens et taubériens de la transformation de Laplace*, Ann. Inst. Henri Poincaré **4** (1966), 49–65. MR **34** #6452.

13. H. Mellin, *Abriss einer allgemeinen und einheitlichen Theorie der asymptotische Reihen*, Wissenschaftliche Vorträge gehalten auf dem 5 Kongress der Skandinav. Mathematiker in Helsingfors, 4–7 Juli 1922, vol. 1, 1922, pp. 1–17.

14. W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Appl. Math., no. 12, Interscience, New York, 1962. MR **27** #2808.

15. L. Schwartz, *Mathematics for the physical sciences*, Hermann, Paris, 1966.

16. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R.I., 1959. MR **21** #5029.

17. E. Wagner, *Taubersche Sätze reeler Art für die Laplace-Transformation*, Math. Nachr. **31** (1966), 153–168. MR **33** #3003.

18. ———, *Ein reeler Tauberschen Satz für die Laplace-Transformation*, Math. Nachr. **36** (1968), 323–331. MR **38** #1436.

19. D. V. Widder, *The Laplace transform*, Princeton Math. Series, vol. 6, Princeton Univ. Press, Princeton, N.J., 1941. MR **3,** 232.

20. ———, *Inversion of a heat transform by use of series*, J. Analyse Math. **18** (1967), 389–413. MR **35** #2088.

21. ———, *An introduction to transform theory*, Pure and Appl. Math., vol. 42, Academic Press, New York, 1971.

22. A. H. Zemanian, *Distribution theory and transform analysis. An introduction to generalized functions, with applications*, McGraw-Hill, New York, 1965. MR **31** #1556.

23. V. Riekstyna, *Generalized asymptotic expansions for a contour integral*, Latvijas Valsts Univ. Zinātn. Raksti **28** (1959), 111–126. MR **23** #A483.

24. ———, *Asymptotic expansions of some integrals and the sums of power series*, Latvian Math. Yearbook **9** (1971), 203–220.

25. E. Wagner, *Taubersche Sätze reeler Art für Integraltransformationen mit Kernen der Form* exp $h(s)t$, Wiss. Z. Univ. Rostock **20** (1971), 313–320.

MATHEMATICAL SCIENCES DEPARTMENT, IBM, T. J. WATSON RESEARCH CENTER, YORKTOWN HEIGHTS, NEW YORK 10598