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LATTICE SUMS OF HOMOGENEOUS FUNCTIONS

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ABSTRACT. An asymptotic estimate for a general lattice sum is derived using Fourier analysis techniques.

1. Let $x_1, \cdots, x_n$ be a basis of $\boldsymbol{R}^n$, $n \geqq 2$, and $L = \boldsymbol{Z}x_1 + \cdots + \boldsymbol{Z}x_n$ the lattice they generate. Let $B(T) = \{x \mid \|x\| \leqq T\}$ and $L(T) = \sum_{L \cap B(T)} 1$. Then it is known that $L(T) = (c_n/d)T^n + O(T^{n(n-1)/(n+1)})$ where c_n = Volume of $B(1)$ and $d = |\mathrm{Det}(x_1, \cdots, x_n)|$. We wish to prove an analogous estimate for a general homogeneous function.

2. Let $f(x)$ be a real-valued homogeneous function on $\boldsymbol{R}^n$, i.e., $f(rx) = r^\alpha f(x)$ for all $r \geqq 0$ with the restriction that $\alpha \geqq 0$. We then define $c_f = \int_{B(1)} f(x)\, dx$ and $L_f(T) = \sum_{x \in L \cap B(T)} f(x)$. We will prove

THEOREM. *If f is homogeneous and $f \in C^{[n/2]+1}(\boldsymbol{R}^n)$ then $L_f(T) = (c_f/d)T^{n+\alpha} + O(T^{n(n-1)/(n+1)+\alpha})$.*

3. The Theorem may be obtained by the methods used in [**1**]. The only thing to prove is that the Fourier transform of $f \cdot X_{B(1)}(x)$ is $O(\|x\|^{-(n+1)/2})$ where $X_{B(1)}(x)$ is the characteristic function of $B(1)$:

$$\widehat{f \cdot X_{B(1)}}(x) = \int_{B(1)} e^{2\pi i(y,x)} f(y)\, dy = \frac{1}{2\pi i\, \|x\|} \int_{\partial B(1)} e^{2\pi i(y,x)} g_1(y, x)\, dy$$

with $g_1(y, x)$ ($[n/2]$)-times differentiable in y (see the Lemma in [**1**] and Lemma 3 in [**3**]). By applying a partition of unity to $\partial B(1)$ and integrating by parts as in [**2**] our estimate follows. The proof of the Theorem is now carried out exactly as in [**1**] using the Poisson summation formula.

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