K-THEORY OF COMMUTATIVE REGULAR RINGS
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Abstract. Pierce's representation of a commutative regular ring as a sheaf of fields is used to compute the $K$-theory of the ring: $K_1$ is units (Robert's Theorem) and $K_2$ is generated by symbols.

Pierce [4] shows how a commutative regular (in the sense of von Neumann) ring $R$ can be represented as the ring of all global sections of a sheaf $R$ of fields over a zero-dimensional compact space $X$. Because of the zero-dimensionality of $X$, data over the stalks of $R$ can be extended to similar global data, and thus much of the theory of $R$ follows immediately from the corresponding theory for fields. This brief note uses this technique to compute the $K$-theory of $R$: $K_0(R)$ is the ring $C(X, \mathbb{Z})$ of continuous functions from $X$ to the integers $\mathbb{Z}$ (this is due to Pierce [4, 16.4, p. 67]); $K_1(R)$ is the group of units of $R$ (this is due to Roberts [5, p. 425]); and $K_2(R)$ is generated by the universal Steinberg symbols. The proofs use only the fact that the $K_i$ are functors of finite type and hence apply to other such functors. This gives new proofs of earlier results on the Brauer group [2, 1.10, p. 117].

We begin by recalling Pierce's general construction: to each commutative ring $R$ is associated a compact, zero-dimensional Hausdorff space $X(R)$ and a sheaf $\mathcal{R}$ of connected rings on $X(R)$ such that $R = \Gamma(X(R), \mathcal{R})$ [4, 4.4, p. 17].

When $R$ is regular, this is just the usual sheaf of local rings on Spec$(R)$ [6, 2.4].

Now suppose $F$ is an abelian group valued functor on the category of commutative rings. Then $F \circ \mathcal{R}$ is a presheaf of abelian groups on $X(R)$. We denote its associated sheaf by $\#F$. Further study of $\#F$ requires the following hypothesis.

Definition (see [1, 1.5, p. 24]). $F$ is of finite type if $F$ commutes with finite products and arbitrary direct limits.

Proposition 1. In the above notation, suppose $F$ is of finite type. Then the group of global sections of $\#F$ is isomorphic to $F(R)$ and for each $x$ in $X(R)$, the stalk $(\#F)_x$ is isomorphic to $F(R_x)$.
Proof. For the first part, see [2, 1.9, p. 117]. For the second part, $(\#F)_x = F \circ \mathcal{R}_x = \text{dir lim } F(\Gamma(U, \mathcal{R})) = F(\text{dir lim } \Gamma(U, \mathcal{R}))$ (since $F$ is of finite type) and $\text{dir lim } \Gamma(U, \mathcal{R}) = R_x$, the direct limit being over open neighborhoods of $x$.

We note the proposition applies to the following functors of finite type: $K_i$, $i=0, 1, 2$, $K_2$ being in the sense of Milnor [3, p. 40], the Picard group, the multiplicative group $G_m$ and the Brauer group. (Regarding the latter, see also [2, 1.10, p. 117].) We also will consider the universal Steinberg symbol functor [3, 11.1, p. 93] defined as follows: for any commutative ring $T$, $Us(T)$ is the multiplicative abelian group with one generator $(a, b)$ for each pair of units $a, b$ of $T$ and relations forcing $(a, b)$ to be bimultiplicative and $(a, b) = 1$ if $a + b = 1$. Clearly $Us$ is a functor of finite type.

Proposition 2. Let $R$ be a commutative regular ring and $F \to G$ a natural transformation of functors of finite type from commutative rings to abelian groups which is an isomorphism when the rings are fields. Then $F(R) \to G(R)$ is also an isomorphism.

Proof. The natural transformation $F \to G$ induces a sheaf morphism $f: \#F \to \#G$. For each $x$ in $X(R)$, $R_x$ is a field [4, 10.3, p. 41] and we have a commutative diagram

$$
\begin{array}{ccc}
\#F_x & \to & \#G_x \\
\downarrow & & \downarrow \\
F(R_x) & \to & G(R_x)
\end{array}
$$

The vertical maps are isomorphisms by Proposition 1 and the lower horizontal map is an isomorphism by hypothesis. Thus the upper map is an isomorphism. Thus $f$, being an isomorphism at each stalk, is an isomorphism and hence induces an isomorphism of global sections. But, by Proposition 1 again, these global sections are $F(R)$ and $G(R)$ respectively.

Corollary 3. Let $R$ be a commutative regular ring. Then:

(a) $K_0R = C(X(R), \mathbb{Z})$,
(b) $K_1R = G_m(R)$,
(c) $K_2R = Us(R)$.

Proof. The transformation $K_0 \to C(X( ), \mathbb{Z})$ is given by the rank homomorphism, the transformation $K_1 \to G_m$ is given by the determinant and the transformation $Us \to K_2$ is given by symbols [3, p. 74]. These are all isomorphisms for fields, the first two being classical and the third a theorem of Matsumoto [3, 11.1, p. 93]. Thus Proposition 2 applies and the corollary follows.

Roberts also computes relative $K_1$ for commutative regular rings [5, p. 425]. We outline another approach to his result.
Lemma 4. Let $R$ be a commutative regular ring, $I$ an ideal of $R$. Then $G_m(R) \to G_m(R/I)$ is onto.

Proof. Let $r$ in $R$ go to a unit of $R/I$, and let $Rr = Re$ where $e$ is idempotent. $I$ is the intersection of the maximal ideals $M$ containing it, and if $I$ is contained in $M$, $r$, being a unit modulo $I$, is not, so $e$ is not in $M$ and hence $1-e$ is. This holds for all $M$ containing $I$, so $1-e$ is in $I$. Let $s=1-e+r$. If $s$ is in the maximal ideal $M$, then if $e \in M$, $r \in M$ so also $1-e \in M$, which is impossible, so $e$ is not in $M$ and $1-e$ is in $M$, hence $r \in M$, hence $e \in M$, again an impossibility. Thus $s$ is in no maximal ideal, hence is a unit congruent to $r$ modulo $I$.

Corollary 5. Let $R$ be a commutative regular ring, $I$ an ideal of $R$. Then $K_1(R, I) = \text{Kernel}(G_m(R) \to G_m(R/I))$.

Proof. We have an exact sequence [3, 6.2, p. 54]

$$K_2R \to K_2R/I \to K_1(R, I) \to K_1R \to K_1R/I$$

which by Corollary 5 becomes

$$K_2(R) \to K_2(R/I) \to K_1(R, I) \to G_m(R) \to G_m(R/I).$$

By Lemma 4 the first map is onto, and the result follows by exactness.

Proposition 1 gives information about rings of continuous functions $C(X, T)$ where $X$ is a compact zero-dimensional topological space and $T$ a commutative ring with the discrete topology. However, a more elementary argument suffices in this special case, which we now record.

Proposition 6. Let $R = C(X, T)$ be as above and let $F$ be a functor of finite type. Then $F(R)$ is isomorphic to $C(X, F(T))$ (here $F(T)$ carries the discrete topology).

Proof. A finite partition of $X$ is a cover of $X$ by finitely many disjoint open subsets. The partitions of $X$ are partially ordered by refinement and $X = \text{proj lim } P$ where $P$ ranges over the partitions of $X$. Thus $C(X, T) = \text{dir lim } C(P, T)$, and $C(P, T)$ is a finite product of copies of $T$. Then since $F$ is of finite type, $F(C(X, T)) = \text{dir lim } F(C(P, T)) = \text{dir lim } C(P, F(T)) = C(X, F(T))$.

By the Stone representation theorem, Boolean rings are of the type treated in Proposition 6 (where $T$ is the field of two elements). Thus the $K$-theory of Boolean rings may be computed:

Corollary 7. Let $R$ be a Boolean ring. Then: $K_0R = C(X(R), Z)$ and $K_1$ and $K_2$ of $R$ are trivial.

Proof. The final assertion follows from Steinberg's result [3, 9.9, p. 75] that $K_2$ of a finite field is trivial.
One may also use Proposition 6 to compute $K$-theory of rings of continuous integer valued functions, and also Brauer groups of such rings and Boolean rings as in [2, 1.12, p. 118].

References