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K-THEORY OF COMMUTATIVE REGULAR RINGS

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ABSTRACT. Pierce's representation of a commutative regular ring as a sheaf of fields is used to compute the K-theory of the ring: K_1 is units (Robert's Theorem) and K_2 is generated by symbols.

Pierce [**4**] shows how a commutative regular (in the sense of von Neumann) ring R can be represented as the ring of all global sections of a sheaf R of fields over a zero-dimensional compact space X. Because of the zero-dimensionality of X, data over the stalks of R can be extended to similar global data, and thus much of the theory of R follows immediately from the corresponding theory for fields. This brief note uses this technique to compute the K-theory of R: $K_0(R)$ is the ring $C(X, Z)$ of continuous functions from X to the integers Z (this is due to Pierce [**4**, 16.4, p. 67]); $K_1(R)$ is the group of units of R (this is due to Roberts [**5**, p. 425]); and $K_2(R)$ is generated by the universal Steinberg symbols. The proofs use only the fact that the K_i are functors of finite type and hence apply to other such functors. This gives new proofs of earlier results on the Brauer group [**2**, 1.10, p. 117].

We begin by recalling Pierce's general construction: to each commutative ring R is associated a compact, zero-dimensional Hausdorff space $X(R)$ and a sheaf $\mathscr{R}$ of connected rings on $X(R)$ such that $R=\Gamma(X(R), \mathscr{R})$ [**4**, 4.4, p. 17].

When R is regular, this is just the usual sheaf of local rings on Spec(R) [**6**, 2.4].

Now suppose F is an abelian group valued functor on the category of commutative rings. Then $F \circ \mathscr{R}$ is a presheaf of abelian groups on $X(R)$. We denote its associated sheaf by $\#F$. Further study of $\#F$ requires the following hypothesis.

DEFINITION (SEE [**1**, 1.5, p. 24]). F is of *finite type* if F commutes with finite products and arbitrary direct limits.

PROPOSITION 1. *In the above notation, suppose F is of finite type. Then the group of global sections of $\#F$ is isomorphic to $F(R)$ and for each x in $X(R)$, the stalk $(\#F)_x$ is isomorphic to $F(R_x)$.*

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PROOF. For the first part, see [**2**, 1.9, p. 117]. For the second part, $(\#F)_x = F \circ \mathscr{R}_x = \operatorname{dir\,lim} F(\Gamma(U, \mathscr{R})) = F(\operatorname{dir\,lim} \Gamma(U, \mathscr{R}))$ (since F is of finite type) and $\operatorname{dir\,lim} \Gamma(U, \mathscr{R}) = R_x$, the direct limit being over open neighborhoods of x.

We note the proposition applies to the following functors of finite type: K_i, $i=0, 1, 2$, K_2 being in the sense of Milnor [**3**, p. 40], the Picard group, the multiplicative group G_m and the Brauer group. (Regarding the latter, see also [**2**, 1.10, p. 117].) We also will consider the universal Steinberg symbol functor [**3**, 11.1, p. 93] defined as follows: for any commutative ring T, $Us(T)$ is the multiplicative abelian group with one generator (a, b) for each pair of units a, b of T and relations forcing (a, b) to be bimultiplicative and $(a, b)=1$ if $a+b=1$. Clearly Us is a functor of finite type.

PROPOSITION 2. *Let R be a commutative regular ring and $F \to G$ a natural transformation of functors of finite type from commutative rings to abelian groups which is an isomorphism when the rings are fields. Then $F(R) \to G(R)$ is also an isomorphism.*

PROOF. The natural transformation $F \to G$ induces a sheaf morphism $f: \#F \to \#G$. For each x in $X(R)$, R_x is a field [**4**, 10.3, p. 41] and we have a commutative diagram

$$\begin{array}{ccc} (\#F)_x & \longrightarrow & (\#G)_x \\ \downarrow & & \downarrow \\ F(R_x) & \longrightarrow & G(R_x) \end{array}$$

The vertical maps are isomorphisms by Proposition 1 and the lower horizontal map is an isomorphism by hypothesis. Thus the upper map is an isomorphism. Thus f, being an isomorphism at each stalk, is an isomorphism and hence induces an isomorphism of global sections. But, by Proposition 1 again, these global sections are $F(R)$ and $G(R)$ respectively.

COROLLARY 3. *Let R be a commutative regular ring. Then*:

(a) $K_0R = C(X(R), Z)$,

(b) $K_1R = G_m(R)$,

(c) $K_2R = Us(R)$.

PROOF. The transformation $K_0 \to C(X(\ ), Z)$ is given by the rank homomorphism, the transformation $K_1 \to G_m$ is given by the determinant and the transformation $Us \to K_2$ is given by symbols [**3**, p. 74]. These are all isomorphisms for fields, the first two being classical and the third a theorem of Matsumoto [**3**, 11.1, p. 93]. Thus Proposition 2 applies and the corollary follows.

Roberts also computes relative K_1 for commutative regular rings [**5**, p. 425]. We outline another approach to his result.

LEMMA 4. *Let R be a commutative regular ring, I an ideal of R. Then $G_m(R) \to G_m(R/I)$ is onto.*

PROOF. Let r in R go to a unit of R/I, and let $Rr = Re$ where e is idempotent. I is the intersection of the maximal ideals M containing it, and if I is contained in M, r, being a unit modulo I, is not, so e is not in M and hence $1-e$ is. This holds for all M containing I, so $1-e$ is in I. Let $s = 1-e+r$. If s is in the maximal ideal M, then if $e \in M$, $r \in M$ so also $1-e \in M$, which is impossible, so e is not in M and $1-e$ is in M, hence $r \in M$, hence $e \in M$, again an impossibility. Thus s is in no maximal ideal, hence is a unit congruent to r modulo I.

COROLLARY 5. *Let R be a commutative regular ring, I an ideal of R. Then $K_1(R, I) = \mathrm{Kernel}(G_m(R) \to G_m(R/I))$.*

PROOF. We have an exact sequence [**3**, 6.2, p. 54]

$$K_2R \to K_2R/I \to K_1(R, I) \to K_1R \to K_1R/I$$

which by Corollary 5 becomes

$$Us(R) \to Us(R/I) \to K_1(R, I) \to G_m(R) \to G_m(R/I).$$

By Lemma 4 the first map is onto, and the result follows by exactness.

Proposition 1 gives information about rings of continuous functions $C(X, T)$ where X is a compact zero-dimensional topological space and T a commutative ring with the discrete topology. However, a more elementary argument suffices in this special case, which we now record.

PROPOSITION 6. *Let $R = C(X, T)$ be as above and let F be a functor of finite type. Then $F(R)$ is isomorphic to $C(X, F(T))$ (here $F(T)$ carries the discrete topology).*

PROOF. A finite partition of X is a cover of X by finitely many disjoint open subsets. The partitions of X are partially ordered by refinement and $X = \mathrm{proj\ lim}\, P$ where P ranges over the partitions of X. Thus $C(X, T) = \mathrm{dir\ lim}\, C(P, T)$, and $C(P, T)$ is a finite product of copies of T. Then since F is of finite type, $F(C(X, T)) = \mathrm{dir\ lim}\, F(C(P, T)) = \mathrm{dir\ lim}\, C(P, F(T)) = C(X, F(T))$.

By the Stone representation theorem, Boolean rings are of the type treated in Proposition 6 (where T is the field of two elements). Thus the K-theory of Boolean rings may be computed:

COROLLARY 7. *Let R be a Boolean ring. Then: $K_0R = C(X(R), Z)$ and K_1 and K_2 of R are trivial.*

PROOF. The final assertion follows from Steinberg's result [**3**, 9.9, p. 75] that K_2 of a finite field is trivial.

One may also use Proposition 6 to compute K-theory of rings of continuous integer valued functions, and also Brauer groups of such rings and Boolean rings as in [**2**, 1.12, p. 118].

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