PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 3, August 1973

ON SPLITTING COTORSION RADICALS

V. S. RAMAMURTHI

Abstract. For a category of modules, the notion, dual to that of a torsion radical, has been called a cotorsion radical. In this paper, the following two properties are examined for a cotorsion radical ρ: (1) If N is a submodule of M and $\rho(M)=M$, then $\rho(N)=N$. (2) The exact sequence $0\to\rho(M)\to M\to M/\rho(M)\to 0$ splits for each module M.

Introduction. The notion of a cotorsion radical for the category $\mathcal{M}_R$ of right R-modules of a ring R, was introduced by Beachy [**2**], as a dual to the notion of torsion radicals due to Maranda [**10**]. A *cotorsion radical* ρ for $\mathcal{M}_R$ is just a subfunctor of the identity on $\mathcal{M}_R$ such that $\rho^2=\rho$ and every epimorphism $M\to N$ induces an epimorphism $\rho(M)\to\rho(N)$. The purpose of this paper is to study the following two properties for a cotorsion radical ρ: (1) The exact sequence $0\to\rho(M)\to M\to M/\rho(M)\to 0$ splits for each $M\in\mathcal{M}_R$. (2) If N, $M\in\mathcal{M}_R$ and $N\subset M$, then $\rho(M)=M$ implies $\rho(N)=N$. A cotorsion radical satisfying (1) (respectively, (2)) is called a *splitting* (respectively, *hereditary*) radical. We give a necessary and sufficient condition for a cotorsion radical to be hereditary and describe all hereditary cotorsion radicals for the category of modules over a semiperfect ring (respectively left Noetherian ring, von Neumann regular ring, integral domain, etc.). For a von Neumann regular ring R, all splitting cotorsion radicals for $\mathcal{M}_R$ are determined and, for a left Noetherian semiprime ring R, it is shown that all hereditary cotorsion radicals for $\mathcal{M}_R$ are splitting. A characterization is given for semiperfect rings R such that all hereditary cotorsion radicals for $\mathcal{M}_R$ are splitting. Our methods also yield a different proof of a result of Bernhardt [**3**], namely, if R is a quasi-Frobenius ring, then all hereditary cotorsion radicals for $\mathcal{M}_R$ are splitting.

Terminology. All rings occurring in this paper are assumed to have identities and all modules to be unitary. Ideal always means two-sided ideal. For all basic definitions see [**9**].

Presented to the Society, December 27, 1972; received by the editors October 12, 1972.

AMS (MOS) subject classifications (1970). Primary 18E40.

Key words and phrases. Torsion radical, cotorsion radical, flat modules, von Neumann regular rings, semiperfect rings, quasi-Frobenius ring.

1. **Hereditary cotorsion radicals.** Let R be a ring and $\mathcal{M}_R$, the category of right R-modules. Beachy [2] has called a subfunctor ρ of the identity functor on $\mathcal{M}_R$ a *cotorsion radical*, if $\rho^2=\rho$ and each epimorphism $M\to N$ in $\mathcal{M}_R$ induces an epimorphism $\rho(M)\to\rho(N)$. A module $M\in\mathcal{M}_R$ is said to be ρ-torsion if $\rho(M)=M$ and ρ-torsion free if $\rho(M)=0$. If ρ is a cotorsion radical for $\mathcal{M}_R$, then $\rho(M)=M\cdot\rho(R)$ for every $M\in\mathcal{M}_R$ (Beachy [2, Proposition 1.3]). Hence ρ is the zero functor (respectively, the identity functor) on $\mathcal{M}_R$ if and only if $\rho(R)=0$ (respectively, $\rho(R)=R$). We call a cotorsion radical ρ, *hereditary*, if any submodule of a ρ-torsion module is ρ-torsion and *splitting* if the exact sequence $0\to\rho(M)\to M\to M/\rho(M)\to 0$ splits for all $M\in\mathcal{M}_R$.

We now consider hereditary cotorsion radicals. The basic result is the following:

(1.1) PROPOSITION. *A cotorsion radical ρ for $\mathcal{M}_R$ is hereditary if and only if $R/\rho(R)$ is flat as a left R-module.*

PROOF. Let $\mathscr{C}$ be the class of ρ-torsion modules and $\mathscr{T}$ the class of ρ-torsion free modules. Then, $(\mathscr{C},\mathscr{T})$ is a torsion theory in the sense of Dickson [5] and it is clear that ρ is hereditary if and only if $\mathscr{C}$ is closed under taking submodules. Now, by (ii) of Proposition 1.3 of Beachy [2], $M\in\mathscr{T}\Leftrightarrow M\cdot\rho(R)=0$. Hence, the image of the inclusion functor $F:\mathcal{M}_{R/\rho(R)}\hookrightarrow\mathcal{M}_R$ is exactly $\mathscr{T}$. Since $\mathcal{M}_{R/\rho(R)}$ is trivially closed under taking injective envelopes and F^{-1} preserves injectives, it follows that $\mathscr{T}$ is closed under taking injective envelopes if and only if F preserves injectives. But the former condition is equivalent to $\mathscr{C}$ being closed under taking submodules (Dickson [5]) and the latter condition is equivalent to $R/\rho(R)$ being flat as a left R-module (Faith [6, Corollary 5A]). Hence the proposition.

This characterization of hereditary cotorsion radicals enables us to identify such radicals in several classes of rings. Our first result in this direction is the following:

(1.2) PROPOSITION. *Let R be a ring for which all cyclic, flat left R-modules are projective. Then, a cotorsion radical ρ for $\mathcal{M}_R$ is hereditary if and only if $\rho(R)$ is a direct summand of R as a left ideal.*

PROOF. By (1.1), ρ is hereditary $\Leftrightarrow R/\rho(R)$ is flat as a left R-module. But, by assumption, all cyclic flat left R-modules are projective. Hence, ρ is hereditary $\Leftrightarrow R/\rho(R)$ is projective as a left R-module $\Leftrightarrow\rho(R)$ is a direct summand of R as a left ideal.

Now we look for examples of rings to which (1.2) is applicable. Bass [1] has called a ring R semiperfect if, for any finitely generated left R-module, there exists a projective left R-module P and an epimorphism

$\pi:P\to M$ with the property: for all submodules P' of P, $P=P'+\ker\pi$ implies $P=P'$. (P is called a projective cover of M.) He has shown that, over a semiperfect ring, any finitely generated flat left R-module is projective. It is well known that any left Noetherian ring also has this property [**8**, p. 61]. Hence, we can apply (1.2) to these two types of rings.

(1.3) COROLLARY. *If R is a semiperfect or a left Noetherian ring, then there is a one-to-one correspondence between hereditary cotorsion radicals for $\mathscr{M}_R$ and ideals of R which are direct summands of R as left ideals.*

PROOF. This follows from the above remarks and the fact that there is a one-to-one correspondence between the cotorsion radicals of any ring and its idempotent ideals, given by $\rho\to\rho(R)$ (Beachy [**2**, Theorem 1.4]).

A ring R is called (von Neumann) regular if, for every $a\in R$, there exists an $x\in R$ such that $a=axa$. It is known that [**9**, p. 134] all modules over a (von Neumann) regular ring are flat. Hence (1.1) immediately yields

(1.4) PROPOSITION. *If R is a (von Neumann) regular ring then all cotorsion radicals for $\mathscr{M}_R$ are hereditary.*

Further, in a (von Neumann) regular ring, all ideals are idempotent. Hence, again by using Theorem 1.4 of Beachy [**2**], we can state

(1.5) COROLLARY. *If R is a (von Neumann) regular ring, then there is a one-to-one correspondence between hereditary cotorsion radicals for $\mathscr{M}_R$ and ideals of R.*

In contrast to the situation for regular rings, there are rings which have very few hereditary cotorsion radicals.

(1.6) PROPOSITION. *If R is a ring, each nonzero ideal of which contains a non-zero-divisor, then $\mathscr{M}_R$ has only two hereditary cotorsion radicals, namely, the zero functor and the identity functor.*

PROOF. Assume that ρ is a hereditary cotorsion radical for $\mathscr{M}_R$ such that $\rho\neq$the zero functor. Then $\rho(R)\neq 0$, so that, by assumption, $\rho(R)$ contains a non-zero-divisor x. Since, by (1.1), $R/\rho(R)$ is flat as a left R-module, there is a left R-homomorphism, $f\colon R\to\rho(R)$ such that $f(x)=x$ [**4**, Proposition 2.2]. This means that $x(1-f(1))=0$ or $1=f(1)$ which gives $\rho(R)=R$. Thus ρ is the identity functor on $\mathscr{M}_R$.

Clearly, the above proposition applies to rings which have no nonzero zero-divisors. Another class of rings to which it applies is the class of prime rings which have ascending chain condition on annihilator right ideals and have no infinite direct sums of right ideals—in particular prime, right Noetherian rings. Because, in such rings each nonzero ideal has nonzero intersection with all nonzero right ideals and hence contains a non-zero-divisor (Goldie [7]).

2. **Splitting cotorsion radicals.**

(2.1) PROPOSITION. *Let ρ be a hereditary cotorsion radical for $\mathscr{M}_R$ and let $\rho(R)$ contain no nonzero nilpotent ideals of R. Then ρ is a splitting radical if and only if $\rho(R)$ is finitely generated as a left ideal.*

PROOF. If ρ is a splitting radical, then $R=\rho(R)\oplus I$ where I is a right ideal of R, so that $R=\rho(R)+RI$ and $\rho(RI)=(RI)\cdot\rho(R)=R(I\cdot\rho(R))=0$, which means that $\rho(R)\cap RI=0$ as ρ is hereditary. Thus $R=\rho(R)\oplus RI$ which shows that $\rho(R)$ is finitely generated as a left ideal. Conversely, if $\rho(R)$ is finitely generated as a left ideal, then $R/\rho(R)$, being a flat left R-module, is also a projective left R-module [**8**, Lemma 2, p. 60]. Thus $R=\rho(R)\oplus L$, for a left ideal L of R. Clearly $R=\rho(R)+LR$ and further $(\rho(R)\cap LR)^2\subset\rho(R)\cdot L=0$ so that, by the assumption on $\rho(R)$, $\rho(R)\cap LR=0$. Thus $R=\rho(R)\oplus LR$ which shows that, for any $M\in\mathscr{M}_R$, $\rho(M)=M.\ \rho(R)$ is a direct summand of M. In other words, ρ is a splitting radical.

The proposition has the following immediate corollaries.

(2.2) COROLLARY. *Let R be a semiprime, left Noetherian ring. Then, all hereditary cotorsion radicals for $\mathscr{M}_R$ are splitting.*

(2.3) COROLLARY. *Let R be a (von Neumann) regular ring. Then, a cotorsion radical ρ for $\mathscr{M}_R$ is splitting if and only if $\rho(R)$ is finitely generated as a left ideal.*

Next we consider the splitting of cotorsion radicals in a semiperfect ring.

(2.4) PROPOSITION. *Let R be a ring in which each cyclic, flat, left R-module is projective. Then all hereditary cotorsion radicals for $\mathscr{M}_R$ are splitting if and only if each ideal of R, which is a direct summand of R as a left ideal, is also a direct summand of R as a right ideal.*

PROOF. Suppose that all hereditary cotorsion radicals for $\mathscr{M}_R$ are splitting. If D is an ideal of R such that $R=D\oplus L$ for a left ideal L, then D is idempotent and hence, by setting $\rho(M)=M\cdot D$, for each $M\in\mathscr{M}_R$, we obtain a cotorsion radical ρ for $\mathscr{M}_R$ such that $\rho(R)=D$, by (Beachy [**2**, Theorem 1.4]). But $R/D\simeq L$ is a projective left R-module, so that, by (1.1), ρ is also hereditary. Hence, by hypothesis, ρ splits, which means that $D=\rho(R)$ is a direct summand of R as a right ideal. To prove the converse, note that, if ρ is a hereditary cotorsion radical for $\mathscr{M}_R$, then, by (1.2), $\rho(R)$ is a direct summand of R as a left ideal and hence, by hypothesis, there is a right ideal I such that $R=\rho(R)\oplus I$. Since $\rho(RI)=RI\cdot\rho(R)=0$, $RI\cap\rho(R)=0$ as ρ is hereditary. Thus $R=RI\oplus\rho(R)$ which shows that $\rho(M)=M.\ \rho(R)$ is a direct summand of M, for each $M\in\mathcal{M}_R$. Thus ρ is a splitting radical.

(2.5) COROLLARY. *If R is a quasi-Frobenius ring, then all hereditary cotorsion radicals for $\mathcal{M}_R$ are splitting.*

PROOF. Recall that a quasi-Frobenius (or QF) ring is a left and right Noetherian ring which is left self-injective (see [8, p. 75]). Hence it suffices to show that in a QF ring R, each ideal which is a direct summand of R as a left ideal, is also a direct summand of R as a right ideal. If D is an ideal in a QF ring R such that $R=D\oplus L$ for a left ideal L, then R/D, as a ring, is left self-injective, since it is injective as a left R-module. It is also a left and right Noetherian ring because R is so. Thus R/D is itself a QF ring and hence is right self-injective (see [8, p. 78]). Now, since R/D is left R-flat, by Corollary 5A of Faith [6], the inclusion functor $\mathcal{M}_{R/D}\hookrightarrow\mathcal{M}_R$ preserves injectives. Thus R/D is injective as a right R-module. But over a QF ring, all injective modules are projective. Hence R/D is a projective right R-module, which means that D is a direct summand of R, as a right ideal.

REMARK. This corollary was proved in [3, Theorem 7] through a different approach.

REFERENCES

1. H. Bass, *Finististic dimension and a homological generalisation of semi-primary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488. MR **28** #1212.

2. J. A. Beachy, *Cotorsion radicals and projective modules*, Bull. Austral. Math. Soc. **5** (1971), 241–253.

3. R. L. Bernhardt, *Splitting hereditary torsion theories over semiperfect rings*, Proc. Amer. Math. Soc. **22** (1969), 681–687. MR **39** #5639.

4. S. U. Chase, *Direct product of modules*, Trans. Amer. Math. Soc. **97** (1960), 457–473. MR **22** #11017.

5. S. E. Dickson, *A torsion theory for abelian categories*, Trans. Amer. Math. Soc. **121** (1966), 223–235. MR **33** #162.

6. C. Faith, *Modules finite over endomorphism ring*, Lecture Notes in Math., no. 246, Springer-Verlag, Berlin, 1972, pp. 146–190.

7. A. W. Goldie, *Semi-prime rings with maximum condition*, Proc. London. Math. Soc. (3) **10** (1960), 201–220. MR **22** #2627.

8. J. P. Jans, *Rings and homology*, Holt, Rinehart and Winston, New York, 1964. MR **29** #1243.

9. J. Lambek, *Lectures on rings and modules*, Blaisdell, Waltham, Mass., 1966. MR **34** #5857.

10. J. M. Maranda, *Injective structures*, Trans. Amer. Math. Soc. **110** (1964), 98–135. MR **29** #1236.

DEPARTMENT OF MATHEMATICS, DE LA SALLE COLLEGE, KARUMATHUR POST, MADURAI DISTRICT, TAMILNADU, INDIA