A VERY SLOWLY CONVERGENT SEQUENCE OF CONTINUOUS FUNCTIONS

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Abstract. A sequence of continuous functions \( f_n : [0, 1] \to (0, 1) \) is constructed, with \( \lim_{n \to \infty} f_n(x) = 0 \) for every \( x \in [0, 1] \), but such that to every unbounded sequence \( \{ \lambda_n \} \) of positive numbers corresponds a point \( x \in [0, 1] \) at which \( \limsup_{n \to \infty} \lambda_n f_n(x) = \infty \).

This may be surprising since the sequence \( \{ f_n \} \) is completely determined by its values on a countable dense set, and since to every countable collection \( \{ S_i \} \) of numerical sequences that tend to 0 there corresponds a sequence \( T \), with \( T(n) \to \infty \), such that \( \lim_{n \to \infty} T(n)S_i(n) = 0 \) for all \( i \).

Each \( x \in K \) (the Cantor set) has a unique representation

\[
x = \sum_{n=1}^{\infty} 3^{-n} a_n(x)
\]

where \( a_n(x) \) is 0 or 2. Define functions \( g_n \in C(K) \) by

\[
g_n(x) = a_1(x) + \cdots + a_n(x) \quad \text{if} \quad a_n(x) = 2,
\]

\[
g_n(x) = 2n - 1 \quad \text{if} \quad a_n(x) = 0.
\]

If \( x \in K \) is fixed, then \( \{ g_n(x) \} \) is a sequence of positive integers in which none occurs twice; thus \( g_n(x) \to \infty \) as \( n \to \infty \). If \( \delta_n > 0 \) and \( \liminf \delta_n = 0 \), there exist integers \( 1 < n_1 < n_2 < \cdots \) such that \( r^2 \delta_{n_r} < 1 \) \( (r = 1, 2, 3, \cdots) \). Choose \( x \in K \), corresponding to \( \{ \delta_n \} \), by specifying that \( a_{n_r}(x) = 2 \) for all \( r \), and that \( a_n(x) = 0 \) otherwise. Then \( \delta_n g_n(x) = 2r\delta_{n_r} < 2/r \) so that \( \liminf \delta_n g_n(x) = 0 \).

To complete the construction, put \( f_n(x) = 1/g_n(x) \) if \( x \in K \), and define \( f_n \) on the rest of \([0, 1]\) by linear interpolation.

Remark 1. These \( f_n \) are piecewise linear. There are polynomials \( P_n \) such that \( f_n < P_n < 2f_n \). This yields a sequence of polynomials with the properties stated in the Abstract.

Remark 2. On the other hand, if \( X \) is compact, \( f_n : X \to (0, \infty) \) is continuous, and \( \lim f_n(x) = 0 \) for every \( x \in X \), then there does exist \( \{ \lambda_n \} \) such
that $\lambda_n \to \infty$ and $\lim \inf_{n \to \infty} \lambda_n f_n(x) = 0$ for every $x \in X$. To see this, put

$$h_n = \min(f_1, \ldots, f_n), \quad m_n = \max_x h_n(x), \quad \lambda_n = 1/\sqrt{m_n}.$$ 

Since $h_n(x) \to 0$ monotonically and $X$ is compact, $m_n \to 0$. To each $x \in X$ corresponds a sequence $\{n_i\}$, $n_i \to \infty$, such that $f_{n_i}(x) = h_{n_i}(x)$. For this $\{n_i\}$,

$$\lambda_{n_i} f_{n_i}(x) = \lambda_{n_i} h_{n_i}(x) \leq \lambda_{n_i} m_{n_i} \to 0 \quad \text{as } i \to \infty.$$