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A VERY SLOWLY CONVERGENT SEQUENCE OF CONTINUOUS FUNCTIONS

WALTER RUDIN[1]

ABSTRACT. A sequence of continuous functions $f_n: [0, 1]\to(0, 1]$ is constructed, with $\lim_{n\to\infty} f_n(x)=0$ for every $x\in[0, 1]$, but such that to every unbounded sequence $\{\lambda_n\}$ of positive numbers corresponds a point $x\in[0, 1]$ at which $\limsup_{n\to\infty} \lambda_n f_n(x)=\infty$.

This may be surprising since the sequence $\{f_n\}$ is completely determined by its values on a countable dense set, and since to every countable collection $\{S_i\}$ of numerical sequences that tend to 0 there corresponds a sequence T, with $T(n)\to\infty$, such that $\lim_{n\to\infty} T(n)S_i(n)=0$ for all i.

Each $x\in K$ (the Cantor set) has a unique representation

$$x=\sum_{n=1}^{\infty} 3^{-n}a_n(x)$$

where $a_n(x)$ is 0 or 2. Define functions $g_n\in C(K)$ by

$$g_n(x)=a_1(x)+\cdots+a_n(x) \quad \text{if } a_n(x)=2,$$
$$g_n(x)=2n-1 \quad \text{if } a_n(x)=0.$$

If $x\in K$ is fixed, then $\{g_n(x)\}$ is a sequence of positive integers in which none occurs twice; thus $g_n(x)\to\infty$ as $n\to\infty$. If $\delta_n>0$ and $\liminf \delta_n=0$, there exist integers $1<n_1<n_2<\cdots$ such that $r^2\,\delta_{n_r}<1$ $(r=1, 2, 3, \cdots)$. Choose $x\in K$, corresponding to $\{\delta_n\}$, by specifying that $a_{n_r}(x)=2$ for all r, and that $a_n(x)=0$ otherwise. Then $\delta_{n_r}g_{n_r}(x)=2r\delta_{n_r}<2/r$ so that $\liminf \delta_n g_n(x)=0$.

To complete the construction, put $f_n(x)=1/g_n(x)$ if $x\in K$, and define f_n on the rest of $[0, 1]$ by linear interpolation.

REMARK 1. These f_n are piecewise linear. There are polynomials P_n such that $f_n<P_n<2f_n$. This yields a sequence of *polynomials* with the properties stated in the Abstract.

REMARK 2. On the other hand, if X is compact, $f_n: X\to(0, \infty)$ is continuous, and $\lim f_n(x)=0$ for every $x\in X$, then there *does* exist $\{\lambda_n\}$ such

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that $\lambda_n \to \infty$ and $\liminf_{n\to\infty} \lambda_n f_n(x) = 0$ for every $x \in X$. To see this, put

$$h_n = \min(f_1, \cdots, f_n), \qquad m_n = \max_x h_n(x), \qquad \lambda_n = 1/\sqrt{m_n}.$$

Since $h_n(x) \to 0$ monotonically and X is compact, $m_n \to 0$. To each $x \in X$ corresponds a sequence $\{n_i\}$, $n_i \to \infty$, such that $f_{n_i}(x) = h_{n_i}(x)$. For this $\{n_i\}$,

$$\lambda_{n_i} f_{n_i}(x) = \lambda_{n_i} h_{n_i}(x) \leqq \lambda_{n_i} m_{n_i} \to 0 \quad \text{as } i \to \infty.$$

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706