

A VERY SLOWLY CONVERGENT SEQUENCE OF CONTINUOUS FUNCTIONS

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ABSTRACT. A sequence of continuous functions $f_n: [0, 1] \rightarrow (0, 1]$ is constructed, with $\lim_{n \rightarrow \infty} f_n(x) = 0$ for every $x \in [0, 1]$, but such that to every unbounded sequence $\{\lambda_n\}$ of positive numbers corresponds a point $x \in [0, 1]$ at which $\limsup_{n \rightarrow \infty} \lambda_n f_n(x) = \infty$.

This may be surprising since the sequence $\{f_n\}$ is completely determined by its values on a countable dense set, and since to every countable collection $\{S_i\}$ of numerical sequences that tend to 0 there corresponds a sequence T , with $T(n) \rightarrow \infty$, such that $\lim_{n \rightarrow \infty} T(n)S_i(n) = 0$ for all i .

Each $x \in K$ (the Cantor set) has a unique representation

$$x = \sum_{n=1}^{\infty} 3^{-n} a_n(x)$$

where $a_n(x)$ is 0 or 2. Define functions $g_n \in C(K)$ by

$$\begin{aligned} g_n(x) &= a_1(x) + \cdots + a_n(x) & \text{if } a_n(x) = 2, \\ g_n(x) &= 2n - 1 & \text{if } a_n(x) = 0. \end{aligned}$$

If $x \in K$ is fixed, then $\{g_n(x)\}$ is a sequence of positive integers in which none occurs twice; thus $g_n(x) \rightarrow \infty$ as $n \rightarrow \infty$. If $\delta_n > 0$ and $\liminf \delta_n = 0$, there exist integers $1 < n_1 < n_2 < \cdots$ such that $r^2 \delta_{n_r} < 1$ ($r = 1, 2, 3, \dots$). Choose $x \in K$, corresponding to $\{\delta_n\}$, by specifying that $a_{n_r}(x) = 2$ for all r , and that $a_n(x) = 0$ otherwise. Then $\delta_{n_r} g_{n_r}(x) = 2r \delta_{n_r} < 2/r$ so that $\liminf \delta_n g_n(x) = 0$.

To complete the construction, put $f_n(x) = 1/g_n(x)$ if $x \in K$, and define f_n on the rest of $[0, 1]$ by linear interpolation.

REMARK 1. These f_n are piecewise linear. There are polynomials P_n such that $f_n < P_n < 2f_n$. This yields a sequence of *polynomials* with the properties stated in the Abstract.

REMARK 2. On the other hand, if X is compact, $f_n: X \rightarrow (0, \infty)$ is continuous, and $\lim_{n \rightarrow \infty} f_n(x) = 0$ for every $x \in X$, then there *does* exist $\{\lambda_n\}$ such

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that $\lambda_n \rightarrow \infty$ and $\liminf_{n \rightarrow \infty} \lambda_n f_n(x) = 0$ for every $x \in X$. To see this, put

$$h_n = \min(f_1, \dots, f_n), \quad m_n = \max_x h_n(x), \quad \lambda_n = 1/\sqrt{m_n}.$$

Since $h_n(x) \rightarrow 0$ monotonically and X is compact, $m_n \rightarrow 0$. To each $x \in X$ corresponds a sequence $\{n_i\}$, $n_i \rightarrow \infty$, such that $f_{n_i}(x) = h_{n_i}(x)$. For this $\{n_i\}$,

$$\lambda_{n_i} f_{n_i}(x) = \lambda_{n_i} h_{n_i}(x) \leq \lambda_{n_i} m_{n_i} \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

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