SHORTER NOTES

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THE NONEXISTENCE OF COMPLEX HAAR SYSTEMS ON NONPLANAR LOCALLY CONNECTED SPACES

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Let \( X \) be a compact Hausdorff space and \( C(X) \) be the complex linear space of complex valued continuous functions defined on \( X \). An \( n \)-dimensional subspace \( L \) of \( C(X) \), \( n \geq 2 \), is called a complex Haar system on \( X \) if every nonzero member of \( L \) has at most \( n - 1 \) zeros. The purpose of this note is to prove

**Theorem 1.** If \( X \) is locally connected, then a necessary and sufficient condition for the existence of a complex Haar system of \( X \) is that \( X \) be imbeddable in the plane.

This affirms a conjecture of J. Overdeck and generalizes the theorem of Schoenberg and Yang [3] in which \( X \) was assumed to be finite polyhedral. To see how their proof extends to the locally connected case, let \( S^2 \) be the 2-sphere, \( K_5 \) and \( K_{3,3} \) be the primitive skew curves (i.e. the complete graph on 5 vertices and the houses and wells configuration), and \( C_1 \) and \( C_2 \) be the Claytor curves as described on the first page of [1].

**Theorem 2 (Claytor [1]).** If \( X \) is a nonplanar Peano continuum, then \( X \) contains a subspace homeomorphic to one of \( S^2 \), \( K_5 \), \( K_{3,3} \), \( C_1 \), or \( C_2 \).

Since a complex Haar system on \( X \) induces one on each of its subspaces, it suffices to show that none of these five spaces admits a complex Haar system. This was done in [3] for \( S^2 \), \( K_5 \), and \( K_{3,3} \). Now observe that \( C_j \) contains a nonempty open set \( U_j \) such that \( C_j - U_j \) is homeomorphic to \( C_j \), \( j = 1 \) or \( 2 \). If there were a complex Haar system on \( C_j \), then \( C_j - U_j \) would be imbeddable in the plane by Lemma 1 of [2]; but this is impossible since \( C_j - U_j \) is homeomorphic to \( C_j \). This proves necessity, and the sufficiency is obvious.

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