A COMBINATORIAL ANALOG OF LYAPUNOV'S THEOREM FOR INFINITESIMALLY GENERATED ATOMIC VECTOR MEASURES

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Abstract. It is shown that the range of a measure obtained by the addition of infinitesimal vectors is convex up to infinitesimal errors.

By a well-known theorem of Lyapunov, the range in $R^n$ of a nonatomic totally finite vector-measure is convex (see [5]). Recent developments in economics (see [3]) have established the need for a similar theorem for a purely atomic measure on a *finite set $A$, where the measure of each atom $a \in A$ is a vector of infinitesimal length. The desired result is a corollary of the following theorem of Steinitz ([7, pp. 167-172]; also see [1, pp. 148-149]): Given any finite collection of vectors in $n$-space $R^n$ with sum 0 and maximum norm $M$, there is an ordering $v_1, v_2, \ldots, v_p$ of those vectors such that the norm of any partial sum $\sum_{i=1}^{p} v_i$ is smaller than $2nM$. (Note that the norm in Steinitz's theorem need not be the Euclidean norm, and if it is, $2nM$ may not be the best constant for dimension $n$.) (See [2] and [4].)

Let $*R$ and $*N$ denote the nonstandard models for the real numbers $R$ and natural numbers $N$ in a fixed enlargement of a structure that contains $R$. (See [6].) We write $a \simeq b$ when $a \in *R$, $b \in *R$ and $a - b$ is infinitesimal, and we denote by $|v|$ the Euclidean distance of any vector $v$ from the origin 0. Fix $n \in *N$, $n$ finite or infinite. If $u$ and $v$ are vectors in $*R^n$, we shall write $u \simeq v$ if $|u - v| \simeq 0$. Recall that a *finite set is a set for which there is an internal one-to-one correspondence with an initial segment of $*N$; such a set has all the "formal" properties of a finite set.

Theorem. Let $A$ be a *finite set, and for each $a \in A$, let $v(a)$ be a vector in $n$-space $*R^n$ with $|v(a)| \simeq 0$. For each internal set $B \subseteq A$, set

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\( S(B) = \sum_{a \in B} v(a) \). Then the following is true:

(i) If \( S(A) \neq 0 \), there is an ordering \( v_1, v_2, \ldots, v_p \) of the set \( \{v(a): a \in A\} \) so that, for each \( j \in \mathbb{N} \) with \( 1 \leq j \leq p \), \( \sum_{i=1}^{j} v_i \equiv \lambda S(A) \) where \( \lambda \in \mathbb{R}^* \).

(ii) Given internal sets \( B \subseteq A \) and \( C \subseteq A \) and given \( \lambda \in \mathbb{R}^* \) with \( 0 < \lambda < 1 \), there is an internal set \( D \subseteq A \) with \( S(D) \equiv \lambda S(B) + (1 - \lambda) S(C) \).

**Proof.** To prove (i), let \( P_L \) denote the projection of \( \mathbb{R}^n \) onto the line \( L \) through 0 and \( S(A) \), and let \( P_H \) denote the projection of \( \mathbb{R}^n \) onto the hyperplane \( H \) perpendicular to \( L \) at 0. The sum of the vectors \( \{P_H(v(a)): a \in A\} \) is 0. Therefore, by Steinitz's theorem, there is an ordering \( v_1, v_2, \ldots, v_p \) of the set \( \{v(a): a \in A\} \) so that, for any \( j \in \mathbb{N} \) with \( 1 \leq j \leq p \),

\[
\left| P_H \sum_{i=1}^{j} v_i \right| = \left| \sum_{i=1}^{j} P_H(v_i) \right| \leq 2n \max_{a \in A} |v(a)| \sim 0,
\]
whence \( \sum_{i=1}^{p} v_i \cong P_L \sum_{i=1}^{p} v_i = \lambda S(A) \) for some \( \lambda \in \mathbb{R}^* \). This proves (i) and more, for given \( \lambda \in \mathbb{R}^* \) with \( 0 < \lambda < 1 \), there is a first \( \omega \in \mathbb{N} \) such that \( |\sum_{i=1}^{\omega} P_L(v_i)| \geq \lambda |S(A)| \), and \( \sum_{i=1}^{\omega} P_L(v_i) \) and \( S(A) \) are on the same side of \( H \). There is thus an \( \omega \in \mathbb{N} \) for which \( \sum_{i=1}^{\omega} P_L(v_i) \cong \lambda S(A) \). We now obtain statement (ii) from this fact by replacing \( A \) with \( B - C \) and \( C - B \) and noting that, for \( 0 < \lambda < 1 \),

\[
\lambda S(B) + (1 - \lambda) S(C) = \lambda S(B - C) + (1 - \lambda) S(C - B) + S(B \cap C).
\]

**References**


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