PROJECTIVE COMPACT DISTRIBUTIVE TOPOLOGICAL LATTICES

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Abstract. In the category of all compact distributive topological lattices and their continuous lattice-homomorphisms, it is shown that every projective object is either zero-dimensional or not I-compact.

By a topological lattice we mean a lattice together with a Hausdorff topology under which the two lattice operations are continuous. All terminologies and notation of lattices and category theory used in this paper are the same as in [2] and in [6], respectively.

Let \( \mathcal{L} \) be a category of topological lattices and their continuous lattice-homomorphisms. By a projective object \( P \) in \( \mathcal{L} \), we mean that, for an onto morphism \( f: A \to B \) and a morphism \( g: P \to B \) in \( \mathcal{L} \), there exists a morphism \( h: P \to A \) in \( \mathcal{L} \) such that \( fh = g \).

Let \( I \) be the unit interval \([0, 1]\) of reals with the usual topology and the usual order structure. For a topological lattice \( L \), if \( L \) is topologically and algebraically isomorphic with a (closed) sublattice of a product of unit intervals, then we say that \( L \) is (I-compact, respectively) I-regular.

Lemma. Let \( \mathcal{L} \) be a category of topological lattices which is closed hereditary and finitely productive. If \( P \) is a connected projective object in \( \mathcal{L} \) then, for every prime ideal \( A \) of \( P \), either \( A \) or \( P \setminus A \) is dense in \( P \).

Proof. We may assume that \( \mathcal{L} \) is nontrivial i.e., \( \mathcal{L} \) has at least one nondegenerate object. Then the two element chain lattice \( 2 = \{0, 1\} \) with the discrete topology is always in \( \mathcal{L} \). Clearly, the closures \( A^- \) and \( (P \setminus A)^- \) are both closed sublattices of \( P \). Let \( Q = (A^- \times \{0\}) \cup ((P \setminus A)^- \times \{1\}) \). Then \( Q \) is a closed sublattice of \( P \times 2 \).

Now let \( j \) be the inclusion of \( Q \) into \( P \times 2 \), and let \( p \) be the projection of \( P \times 2 \) onto \( P \). Then \( pj: Q \to P \) is onto. Since \( P \) is projective, for \( pj \) and the identity \( i \) of \( P \), there exists a morphism \( f: P \to Q \) in \( \mathcal{L} \) such that \( pjf = i \).

Since \( P \) is connected, either \( f(P) \subseteq A^- \times \{0\} \) or \( f(P) \subseteq (P \setminus A)^- \times \{1\} \). If

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let \( f(P) \subseteq A^- \times \{0\} \), then \( P = f^{-1}(A^- \times \{0\}) \). On the other hand, we can show that \( f^{-1}(A^- \times \{0\}) = A^- \). It suffices to show that \( f^{-1}(A^- \times \{0\}) \subseteq A^- \). Let \( x \in f^{-1}(A^- \times \{0\}) \). Suppose that \( f(x) = (y, 0) \in A^- \times \{0\} \). Since \( pyf = i \), we have \( x = y \). Thus \( x \in A^- \). Hence \( A \) is dense in \( P \). Similarly, for the case that \( f(P) \subseteq (P \setminus A) \times \{1\} \), \( P \setminus A \) is dense in \( P \).

**Remark.** With a few additional conditions to those of the above lemma, it can be generalized to some other Hausdorff topological algebras of finite type as follows:

Let \( \mathcal{A} \) be a category of Hausdorff topological algebras of the same finite type which is closed hereditary and finitely productive, and let \( P \) be connected projective in \( \mathcal{A} \). If

(i) the two point algebra \( 2 \) with the discrete topology is in \( \mathcal{A} \),

(ii) \( A \) and \( P \setminus A \) are both subalgebras of \( P \) and \( Q = (A^- \times \{0\}) \cup ((P \setminus A)^- \times \{1\}) \) is a closed subalgebra of \( P \times 2 \) then either \( A \) or \( P \setminus A \) is dense in \( P \).

For example, in the case of Hausdorff topological spaces (as trivial algebras) (i) and (ii) are always true and, in the case of topological semigroups, if \( A \) is a prime ideal of \( P \) and the two point meet semilattice with discrete topology is in \( \mathcal{A} \), then (i) and (ii) are always true.

**Theorem.** Let \( \mathcal{L} \) be a category of topological distributive lattices which is closed hereditary and finitely productive. Then every projective object in \( \mathcal{L} \) is either totally disconnected or not \( I \)-regular.

**Proof.** Let \( P \) be projective in \( \mathcal{L} \). Suppose that \( P \) is not totally disconnected. Then we have a connected component \( C \) of \( P \) with more than two points, and it is a closed convex sublattice of \( P \) [4]. Let \( J = [\alpha, \beta] \) be a nondegenerate closed interval of \( C \). Then \( J \) is also a closed interval in \( P \), which is connected since \( C \) is. Further, it is easy to see that the map \( f : P \to J = [\alpha, \beta] \) defined by \( f(x) = \alpha \lor (x \land \beta) \) is a retraction. Hence \( J \) is also projective in \( \mathcal{L} \). Now we show that \( J \) does not have a nonconstant continuous lattice-homomorphism from \( J \) into \( I \). Indeed, if \( g : J \to I \) is a nonconstant continuous lattice-homomorphism, then \( g(J) = [r, s] \subseteq I \) with \( r < s \). It is easy to see that \( f^{-1}([r, t]) \) \((r < t < s)\) is a closed prime ideal of \( J \), and it is neither dense in \( J \) nor is its complement dense in \( J \). This is a contradiction of the lemma.

**Corollary 1.** Let \( \mathcal{D} \) be the category of all compact distributive lattices. Then every projective lattice in \( \mathcal{D} \) is either zero-dimensional or not \( I \)-compact.

It is known [7] that if \( L \) is a compact distributive lattice then \( L \) is \( I \)-compact iff \( L \) is completely distributive. Hence by the theorem every projective lattice in the category of all compact completely distributive lattices and their continuous lattice-homomorphisms is zero-dimensional.
It is shown [3] that, in the category of all zero-dimensional compact
distributive lattices, \( P \) is projective iff \( P \) is a retract of the residually finite
completion of a free distributive lattice.

Hence we have the following:

**Corollary 2.** Let \( \mathcal{CD} \) be the category of all compact completely
distributive lattices. Then every projective lattice in \( \mathcal{CD} \) is a retract of the
residually finite completion of a free distributive lattice.

**Remark.** It is known [5] that there actually exists a compact distribu-
tive lattice which is not \( I \)-compact. However, the author does not know
whether a projective one which is not \( I \)-compact exists in \( \mathcal{D} \). If such a
projective \( P \) exists in \( \mathcal{D} \), then \( P \) must have the following properties
(i)-(iii):

(i) \( P \) has a nondegenerate connected retract which has no nonconstant
continuos lattice-homomorphism into \( I \).

(ii) \( P/\rho \), where \( x \rho y \) iff \( x \) and \( y \) belong to the same connected component
of \( P \), is projective in \( \mathcal{D} \).

(iii) If \( P \) is connected then, for any upper (or lower) bound \( x \) of a non-
empty open subset of \( P \), \( x \vee P \) (or \( x \wedge P \) respectively) has void interior.

**References**

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