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TRANSFORMATION GROUPS OF AUTOMORPHISMS OF $C(X,G)$

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ABSTRACT. If (X, T, π) is a transformation group with locally compact phase group T, there is a standard way to induce a transformation group on $C(X, Y)$ endowed with the compact-open topology, where Y is a uniform space. In this paper, we consider the case where Y is a topological group G. The reverse construction under certain conditions is also considered.

If (X, T, π) is a transformation group with locally compact phase group T, there is a standard way [3, 1.68] to induce a transformation group on $C(X, Y)$ endowed with the compact-open topology, where Y is a uniform space. England and Lanier [1] have applied this to the case where Y is the space R of all real numbers and have shown that under certain conditions this construction can be reversed and that many dynamical statements about (X, T, π) can be faithfully reflected in $C(X)$. The purpose of this paper is to show that many results obtained in [1] can also be obtained under a more general setting.

For a topological space X and a topological group G, let $C(X, G)$ be the topological group of all continuous functions from X into G endowed with the compact-open topology and with the pointwise multiplication. The identity element of the group $C(X, G)$ is denoted by I_0 which maps every x in X into e, the identity element of G. For each p in X, let $C_p(X, G)$ be composed of functions f such that $f(p)=e$. The groups $C(X, G)$ and $C_p(X, G)$ are discussed in [6]. It is pointed out there that G is isomorphic to $C(X, G)/C_p(X, G)$ for each p.

In our investigation we adopt some notations which are similar to those used in [1] and [2] whenever possible. Throughout this note all transformation groups are supposed to have a locally compact phase group, and the letter G will stand for an arbitrarily chosen but fixed topological group. All topological spaces considered here are Hausdorff. For notations and definitions not given, the reader is referred to [3] and [5].

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Suppose now that (X, T, π) is a transformation group. Define

$$\rho: C(X, G) \times T \to C(X, G)$$

by $\rho(f, t)(x) = f(\pi(x, t^{-1}))$ as in [**3**, 1.68]; then $(C(X, G), T, \rho)$ is also a transformation group, called the induced functional transformation group.

For $f \in C(X, G)$, we let $Z(f) = \{x \in X : f(x) = e\}$.

DEFINITION 1. We shall call a pair (X, G) of a topological space X and a topological group G an S-pair if, for each closed subset C of X and $x \notin C$, there exists an f in $C(X, G)$ such that $Z(f) \supset C$ and $f(x) \neq e$.

REMARK 2. S-pairs of topological spaces and topological groups are abundant. Professor R. V. Fuller has pointed out that if (X, G) is an S-pair, then X is completely regular.

For the remainder of this paper, all topological spaces X in $C(X, G)$ will be assumed to be those such that (X, G) are S-pairs. It is clear that (X, T, π) is effective if and only if $(C(X, G), T, \rho)$ is effective, and that, for each $t \in T$, ρ^t is a (topological) isomorphism of $C(X, G)$ onto $C(X, G)$ keeping constant functions invariant.

For each $p \in X$, let $M_p = C_p(X, G) = \{f \in C(X, G) : f(p) = e\}$ and $O_p = \{f \in C(X, G) : Z(f)$ is a neighborhood of $p\}$. Then O_p is a subgroup of $C(X, G)$ contained in M_p, and $M_p \neq M_q$ if $p \neq q$.

LEMMA 3. *Let (X, T, π) be a transformation group and let $(C(X, G), T, \rho)$ be the induced functional transformation group. Then, for each $t \in T$, $\rho^t(M_p) = M_{\pi(p,t)}$.*

PROOF. The proof follows easily from the definitions.

DEFINITION 4. (1) A normal subgroup M of $C(X, G)$ is called an F-normal subgroup if $\{Z(f); f \in M\}$ has the finite intersection property.

(2) A pair (X, G) of a topological space X and a topological group G is said to be a Q-pair if whenever M is an F-normal subgroup of $C(X, G)$ such that $C(X, G)/M$ isomorphic to G by an isomorphism ϕ such that $\phi(c_y M) = y$, then $\bigcap_{f \in M} Z(f) \neq \phi$, where c_y is the constant function which maps X into y.

Note that if X can be embedded into G as a subspace of G, then (X, G) is a Q-pair. To see this, we may identify X as a subspace of G and let $i: X \to G$ be the inclusion map. If M is an F-normal subgroup of $C(X, G)$ such that $C(X, G)/M$ is isomorphic to G by an isomophism ϕ such that $\phi(c_y M) = y$, then there is a unique $t \in G$ such that $\phi(iM) = t$. Thus $ic_{t^{-1}} \in M$. Now $Z(ic_{t^{-1}}) = \{t\}$, and M is an F-normal subgroup, hence $\bigcap_{f \in M} Z(f) = \{t\}$. Since every completely regular space can be embedded as a closed subspace of complex topological linear space [**4**, 8.21], every completely regular space has a topological group such that (X, G) is a Q-pair.

REMARK 5. Suppose that (X, G) is a Q-pair. Then the only F-normal subgroups M of $C(X, G)$ such that $C(X, G)/M$ is isomorphic to G by an isomorphism ϕ such that $\phi(c_y M)=y$ are of the form M_p, $p \in X$.

PROOF. Indeed each M_p has the stated property. If M is an F-normal subgroup of $C(X, G)$ with the stated property, then there is $p \in \bigcap_{f\in M} Z(f)$, and M is a normal subgroup of $C(X, G)$ contained in M_p. Since $G= C(X, G)/M=C(X, G)/M/M_p/M=G/M_p/M$, hence $M_p=M$.

If (X, T, π) is a transformation group and if M is an F-normal subgroup of $C(X, G)$, it is clear that $\rho^t(M)$ is also an F-normal subgroup of $C(X, G)$ for each $t \in T$. We say that $(C(X, G), T, \phi)$ is a transformation group of group isomorphisms on $C(X, G)$ keeping constant functions invariant if each ϕ^t is an isomorphism of $C(X, G)$ onto $C(X, G)$, maps each F-normal subgroup into an F-normal subgroup and maps each constant function into itself.

THEOREM 6. *If X is locally compact, (X, G) is a Q-pair, and if $(C(X, G), T, \phi)$ is a transformation group of group isomorphisms on $C(X, G)$ keeping constant functions invariant, then there exists a transformation group (X, T, π) such that the induced functional transformation group $(C(X, G), T, \rho)$ is $(C(X, G), T, \phi)$.*

PROOF. Define $\pi X\times T\to X$ by $\pi(p, t)=q$ if and only if $\phi^t(M_p)=M_q$.

CLAIM 1. π is well defined.

PROOF OF CLAIM 1. For each $t \in T$, $\phi^t(M_p)$ is an F-normal subgroup of $C(X, G)$ for $p \in X$. Moreover, it is easy to see that $C(X, G)/\phi^t(M_p)$ is isomorphic to G by an isomorphism ϕ such that $\phi(c_y M)=y$. Hence Remark 5 implies that there is a unique $q \in X$ such that $\phi^t(M_p)=M_q$. Hence π is well defined.

CLAIM 2. (X, T, π) is a transformation group.

PROOF OF CLAIM 2. Verification of the identity and homorphism axioms are routine, only continuity of π needed to be shown.

Let C be a closed subset of X, and let (x_α, t_α), $\alpha \in A$, be a net in $\pi^{-1}(C)$ such that $(x_\alpha, t_\alpha)\to(x, t)$. We want to show $(x, t) \in \pi^{-1}(C)$. Suppose $(x, t) \notin \pi^{-1}(C)$, then $\pi(x, t) \notin C$. Let $f \in C(X, G)$ such that $Z(f)\supset C$ but $f(\pi(x, t))\neq e$. Then $f \notin M_{\pi(x,t)}$. Note that $f \in M_{\pi(y,z)}$ if and only if $\phi^{z^{-1}} f \in M_y$. Now $\phi t_\alpha^{-1} f\to\phi t^{-1} f$ in $C(X, G)$ since ϕ is continuous. Since X is locally compact, the compact-open topology for $C(X, G)$ is jointly continuous. Hence $\phi t_\alpha^{-1} f$ converges continuously to $\phi^{t^{-1}} f$ [4, p. 241], so $\phi^{t_\alpha^{-1}} f(x_\alpha)\to \phi t^{-1} f(x)$ in G. But $\pi(x_\alpha, t_\alpha) \in C$ for each $\alpha \in A$, hence $f \in M_{\pi(x_\alpha, t_\alpha)}$, i.e. $\phi^{t_\alpha^{-1}} f \in M_{x_\alpha}$ for each $\alpha \in A$. Hence $\phi^{t_\alpha^{-1}} f(x_\alpha)=e$. This implies that $\phi^{t^{-1}} f(x)=e$. Thus $\phi^{t^{-1}} f \in M_x$, i.e., $f \in M_{\pi(x,t)}$ which contradicts with $f \notin M_{\pi(x,t)}$. This proves Claim 2.

We now return to the proof of Theorem 6. Since $\rho^t(M_p)=M_{\pi(p,t)}=\phi^t(M_p)$ for each $p\in X$ and each $t\in T$, we only need to show that, for each $t\in T$, $\phi^t(f)=\rho^t(f)$ for $f\in C(X,G)$. Suppose there is a $f\in C(X,G)$, a $t\in T$ and a $p\in X$ such that $\phi^t(f)(p)\neq\rho^t(f)(p)$. Since ϕ^t and ρ^t are group isomorphisms keeping constant functions invariant, we may assume that $\phi^t f(p)=e$ but $\rho^t f(p)\neq e$. Then $\phi^t f\in M_p$ but $\rho^t f\notin M_p$. However, $f=\phi^{t^{-1}}(\phi^t f)\in\phi^{t^{-1}}(M_p)=M_{\pi(p,t^{-1})}$ and $f=\rho^{t^{-1}}(\rho^{t^{-1}}f)\notin\rho^{t^{-1}}(M_p)=M_{\pi(p,t^{-1})}$ a contradiction. Hence the theorem is proved.

If $f\in C(X,G)$ such that $Z(f)\neq\phi$, let $[f]=\{gh^\varepsilon h: g, h\in C(X,G)$ such that $Z(g)\supset Z(f)$, $Z(h)\supset Z(f)$, and $\varepsilon=1$ or $-1\}$. Then $f\in[f]$ and $[f]$ is a subgroup of $C(X,G)$ contained in M_x for each $x\in Z(f)$. Since ρ^t is a group isomorphism for each $t\in T$, it is easy to see that $\rho^t([f])=[\rho^t f]$.

For each subgroup M of $C(X,G)$ and each subset S of T, let $J(M;S)=\bigcap_{t\in S}\rho^t(M)$. Note that $J(M;S)$ is again a subgroup of $C(X,G)$ and that, if $S_1\subset S_2$, $J(M;S_2)\subset J(M;S_1)$.

The following theorem is the key theorem to our investigation for the remainder of this paper.

THEOREM 7. *Suppose that $f\in M_p$ and that A and B are any subset of T. Then $J([f];A)\subset J(M_p;B)$ if and only if $\pi(p,B)\subset \mathrm{cl}(\pi(Z(f),A))$.*

PROOF. Suppose $\pi(p,B)\subset\mathrm{cl}(\pi(Z(f),A))$, and let $g\in J([f];A)=\bigcap_{t\in A}\rho^t([f])=\bigcap_{t\in A}[\rho^t f]$. Then $Z(g)\supset Z(\rho^t f)=\pi(Z(f),t)$ for each $t\in A$. Hence $Z(g)\supset\mathrm{cl}(\pi(Z(f),A))$ since $Z(g)$ is closed, and we have $Z(g)\supset\pi(p,B)$. This shows that $g\in M_{\pi(p,t)}=\rho^t(M_p)$ for each $t\in B$. Hence $g\in\bigcap_{t\in B}\rho^t(M_p)=J(M_p;B)$.

Conversely, assume that $J([f];A)\subset J(M_p;B)$, and let $q\notin\mathrm{cl}(\pi(Zf),A))$. Then there is $g\in C(X,G)$ with $Z(g)\supset\mathrm{cl}(\pi(Z(f),A))$ but $q\notin Z(g)$. Since $Z(\rho^t f)=\pi(Z(f),t)\subset\pi(Z(f),A)$ for each $t\in A$, we have that $Z(g)\supset Z(\rho^t f)$ for each $t\in A$. If $h=\rho^t f$, $[h]=[\rho^t f]=\rho^t([f])$. Let $k=gh^{-1}$. Then $Z(k)\supset Z(h)$ and $g=kh=khI_0\in[h]=\rho^t([f])$. Hence $g\in J([f];A)\subset J(M_p;B)$. Thus $g\in\rho^t(M_p)=M_{\pi(p,t)}$ for each $t\in B$, and $Z(g)\supset\pi(p,B)$. This implies that $q\notin\pi(p,B)$. Hence $\pi(p,B)\supset\mathrm{cl}(\pi(Z(f);A))$.

DEFINITION 8. (1) Let $\mathscr{A}$ be a class of subsets of T. The elements of $\mathscr{A}$ are called admissible sets.

(2) The normal subgroup M_p is said to be periodic under T if there exists a compact subset K of T such that $J(M_p;K)\subset J(M_p;T)$.

(3) The normal subgroup M_p is said to be A-recursive under T if for each $f\in O_p$ there is an admissible set A in $\mathscr{A}$ such that $[f]\subset J(M_p;A)$. In particular, if $\mathscr{A}$ is the class of left syndetic subsets of T, and if, for each $f\in O_p$, there is an A in $\mathscr{A}$ such that $[f]\subset J(M_p;A)$, then we say that M_p is almost periodic under T.

REMARK 9. If M_p is periodic under T, then M_p is almost periodic under T. The converse fails.

PROOF. Let $f \in O_p$, and let K be a compact subset of T such that $J(M_p; K) \subset J(M_p; T)$. Then $J([f]; K) \subset J(M_p; T)$. Hence, by Theorem 7, $\pi(p, t) \subset \pi(Z(f), K)$. Since $Z(f)$ is a neighborhood of p, there is a left syndetic A of T such that $\pi(p, A) \subset Z(f) = \pi(Z(f), e)$ [**3**, 4.02]. Hence $[f] = J([f]; \{e\}) \subset J(M_p; A)$, and M_p almost periodic.

The failure of the converse follows from the following two theorems and well-known facts.

Using Theorem 7, we are now able to state the following two theorems. The proofs are easily adapted mutatis mutandis from the proofs of Theorems 7 and 9 of [**1**].

THEOREM 10. *M_p is periodic under T if and only if p is periodic under T.*

THEOREM 11. *M_p is recursive under T if and only if p is recursive under T.*

Now by replacing the term "admissible set" by an appropriate phrase, as in [**3**, 3.38], we may define all of the classical recursive properties for the normal subgroups M_p in $C(X, G)$, and Theorem 11 indicates that many dynamical statements about (X, T, π) can be faithfully reflected in $(C(X, G), T, \rho)$.

As an application to what we discussed so far, we have the following rather easy remark.

PROPOSITION. *Suppose that (X, T, π) is recursive on X and that T is abelian. If $f \in C(X, G)$ is uniformly continuous, then $(C(X, G), T, p)$ is recursive on the coset fN of some subgroup N of $C(X, G)$, where the left uniformity for G is taken as the uniformity for G.*

PROOF. Let $W(C, V_L) = \{(h, k): h(x)^{-1}k(x) \in V, x \in C\}$ be a basic member of the compact-index uniformity of $C(X, G)$ by the left uniformity of G, where C is a compact subset of X and V an open symmetric neighborhood of the identity e_G in G. By the uniform continuity of f, there is a symmetric index α of X such that if x and y are in X with $(x, y) \in \alpha$, we have $f(x) \in f(y)V$. Let A be an admissible subset of T such that $\pi(x, A) \subset \alpha[x]$ for each $x \in C$. If g is a function in $C(X, G)$ such that $a[C] \subset Z(g)$ then $N = [g]$ is a subgroup of $C(X, G)$.

We now show that $(C(X, G), T, \rho)$ is recursive on fN. First we note that M_p is recursive under T by Theorem 11 for each $p \in C$, thus $[g] \subset J(M_p; A')$ for some admissible subset A' of T which may be taken to be A since $\pi(p, A) \subset Z(g)$ if and only if $[g] = J([g], \{e\}) \subset J(M_p; A)$ by Theorem 7. Hence $fh(\pi(p, t)) = f(\pi(p, t))$ for each $h \in N$, $p \in C$ and $t \in A$. Now, for $y \in A$, $\rho(fh, t^{-1})(y) = fh(\pi(y, t)) = f(\pi(y, t)) \in f(\alpha[y]) \subset f(y)V = fh(y)V$.

Hence $\rho(fh, A^{-1}) \subset W(C, V_L)[fh]$ for each $h \in N$, and the proof is completed.

In concluding the paper, we show that Theorem 10 of [**1**] may not hold if we simply replace $C(X)$ by $C(X, G)$ there.

THEOREM 12. *In order that the transformation group* (X, T, π) *be minimal it is sufficient that the only normal subgroups in* $C(X, G)$ *which are invariant under* $H=\{\rho^t : t \in T\}$ *are* $\{I_0\}$ *and* $C(X, G)$. *This sufficient condition is not a necessary condition.*

PROOF. Let $p \in X$. By definition, $J(M_p; T)$ is invariant under H. Let f be the constant map such that $f(x)=a\neq e$ for each $x \in X$. Then $f \notin J(M_p; T)$. Hence $J(M_p; T)\neq C(X, G)$. This implies $J(M_p; T)=\{I_0\}$. Suppose $\mathrm{cl}(\pi(p, T))\neq X$. Then there is $f \in C(X, G)$ such that $f\neq I_0$ and $Z(f)\supset \mathrm{cl}(\pi(p, T))$. This implies by Theorem 7 that $[f]\subset J(M_p; T)$. Hence, in particular, $f \in J(M_p; T)$ which is a contradiction.

The following example shows that the stated condition is not a necessary condition.

EXAMPLE. Let G be the additive group of integers modulo 2 with the discrete topology. The transformation group (G, G, π), where π is the usual addition modulo 2, is minimal. Now $C(G, G)=\{I_0, f_1, f_2, f_3\}$, where f_1 is the function which maps G into 1, f_2 is the function which maps 1 into 1 and 0 into 0, and f_3 is the one which maps 0 into 1 and 1 into 0. Then $M=\{I_0, f_1\}$ is a subgroup of $C(G, G)$ which is invariant under $H=\{\rho^t : t \in G\}$.

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