SPHERES WHICH ARE LOOP SPACES mod $p$

CLARENCE WILKERSO1

Abstract. If $S_n^{2n-1}$ has a loop space structure, then $n|p-1$.

Which spheres have $H$-structures or loop structures is known, see [1]. The $H$-space structure mod $p$ version was answered by [2] and [10]; and recently, Sullivan [10] has shown the following for odd prime $p$:

Theorem 1. $S_n^{2n-1}$ has a loop space structure if and only if $n|p-1$.

The purpose of this note is to present a proof of the necessity that $n|p-1$, via calculations with the Adams operations in $K$-theory, as opposed to the usual secondary cohomology operations [9], [8]. For an account of the Adams operations, see [3] and [4]. For other applications of the Adams operations, see [5], [6], [7], and [11]. Our notation will follow [10]. That is,

$$Z_{(p)} = \{\text{integers localized at } p\};$$

$$= \{\text{rationals with denominator prime to } p\};$$

$X_{(p)}$ is the $p$-localization of $X$, for $X$ a simple CW complex.

Definition 1. If $K$ is a filtered algebra over $Z_{(p)}$, $K$ has a $\{\psi^k\}$ action if and only if there exists a sequence of filtered algebra homomorphisms $\{\psi^k\}$ satisfying the properties of the Adams operations as listed in [4]. That is,

(i) $\psi^p x = x^p \mod p$,
(ii) $\psi^p \psi^q = \psi^q \psi^p$,
(iii) $\psi^k x - k^n x \in K_{n+1}$, if $x \in K_n$,
(iv) if $x \in K_n$, $\exists v_{n+i} \in K_{n+i(p-1)}$ so that $\psi^p x = \sum_i p^{n-i} v_{n+i}$ where $0 \leq i \leq n$.

In particular, $K(X) \otimes Z_{(p)}$ has a $\{\psi^k\}$ action, for any finite CW complex $X$, with $K_n = \ker\{i: X_{2n-1} \rightarrow X\}^*$.

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THEOREM 2. If $K=\mathbb{Z}[x_n]/(x_n^{p+1})$ has a $\{\psi^k\}$ action, then $n|p-1$. Here $K$ is a truncated polynomial algebra on one generator $x_n$ of degree $2n$, and $K$ is filtered by even degrees.

Thus, if $S^{2n-1}_{(p)}=\Omega BS^{2n-1}_{(p)}$, then for an $N$-skeleton, $K((BS_{(p)}^{2n-1})^N)\otimes \mathbb{Z}_{(p)}$ will be a truncated polynomial algebra on one generator [11]. If $N$ is large enough, we can truncate at level $(p+1)$. This preserves the $\{\psi^k\}$ action; hence Theorem 2 implies $n|p-1$, and the necessary part of Theorem 1 is completed.

PROOF OF THEOREM 2.

Notation. $r(i,j)=1$ if $ij \equiv 0 \mod p-1$, =0 otherwise.

LEMMA 1. $\sum_{i<p} r(m,j)=\text{GCD}(m,p-1)=m/(p-1)$.

LEMMA 2. Given $a$, a positive integer, there exists $q$, an integer, so that for all $\beta \leq a+1$, $q^\beta -1 \equiv 0 \mod p^a$ if and only if (a) $p-1|k$ and (b) $p^{\beta-1}|k$.

PROOF. Choose $q$ a generator of the units in $\mathbb{Z}/p^a+1$.

DEFINITION 2. If there is a $\{\psi^k\}$ action on $\mathbb{Z}_{(p)}[x_n]/(x_n^{p+1})$, denote by $\langle \psi^k x, x^s \rangle$ the coefficient of $x^s$ in $\psi^k x$.

LEMMA 3. $\langle \psi^k x, x^s \rangle=0 \mod p^{n-1}(\sum_i r(m,i)(x_i+1))$ where $n=mp^a$ and $p^i|m$ and $i \leq p$. Denote this exponent by $N_i$.

PROOF. The proof of Lemma 3 is by induction on $i$. For $i=1$, the statement of Lemma 3 is $\langle \psi^p x, x \rangle=0 \mod p^n$. This is true by property (iii). Choose a $q$ as in Lemma 2, and let $i<p$.

By our notation, $\psi^p x = \sum_{s \geq 1} \langle \psi^p x, x^s \rangle x^s$, $\psi^k x = \sum_{s \geq 1} \langle \psi^k x, x^s \rangle x^s$. Therefore,

$$\psi^p \psi^x = \sum_{s \geq 1} \langle \psi^x, x^s \rangle \langle \psi^p x, x^s \rangle = \sum_{s \geq 1} \left( \sum_{k \geq 1} \langle \psi^x, x^k \rangle \langle \psi^p x, x^k \rangle \right)^s \langle \psi^x, x^s \rangle.$$  

Similarly,

$$\psi^q \psi^x = \sum_{s \geq 1} \left( \sum_{k \geq 1} \langle \psi^x, x^k \rangle \langle \psi^q x, x^k \rangle \right)^s \langle \psi^q x, x^s \rangle.$$  

The inductive hypothesis is that $\langle \psi^p x, x^k \rangle=0 \mod p^{N_k}$, for $k \leq i<p$. In particular, $\langle \psi^p x, x^k \rangle=0 \mod p^{N_i}$, for $k \leq i<p$. Looking only at $x^{i+1}$ in the above expansions we see that

$$\langle \psi^p \psi^x, x^{i+1} \rangle = \langle \psi^p x, x^{i+1} \rangle \langle \psi^p x, x \rangle^{i+1} \mod p^{N_i},$$  

$$\langle \psi^q \psi^x, x^{i+1} \rangle = \langle \psi^q x, x^{i+1} \rangle \langle \psi^q x, x \rangle \mod p^{N_i}.$$  

But $\psi^q \psi^p - \psi^q \psi^p = 0$ by (ii). So

$$\langle \psi^p x, x^{i+1} \rangle \langle \psi^x, x \rangle - \langle \psi^q x, x \rangle = 0 \mod p^{N_i}. $$
Since \( \langle \psi^n x, x \rangle = q^n \) by property (iii), we have \( q^n(q^{i-1} - 1) \langle \psi^n x, x^{i+1} \rangle = 0 \mod p^{N_i} \). By Lemma 2, \( q^{i-1} - 1 = 0 \mod p^{r(i, n)} \). But \( r(i, n) = r(i, m) \), so \( \langle \psi^n x, x^{i+1} \rangle = 0 \mod p^{N_i - r(i, m)(x+1)} \). Since \( N_{i+1} = N_i - r(i, m)(x+1) \), we have established the lemma.

The proof of Theorem 2 is completed by observing that for \( i = p \), Lemma 3 gives \( \langle \psi^n x, x^p \rangle = 0 \mod p^{n - (m, p-1)(x+1)} \), and \( n - (m, p-1)(x+1) \geq 1 \) if \( x > 0 \). Also, \( m - (m, p-1) \geq 1 \) if \( m \nmid p-1 \). Since by property (i) of the Adams operations, \( \langle \psi^n x, x^p \rangle \neq 0 \mod p, x = 0 \) and \( m \nmid p-1 \).

Remark. An alternate statement of these results can be given with a more general formulation:

Theorem 3. If \( Y \) is a CW space such that \( K(Y) \otimes \mathbb{Z}_p = R \otimes S \) with \( R \cong \mathbb{Z}_p[x]/(x^p) \) for \( r > p \), \( x \in K_n \), \( x \notin K_{n+1} \) and \( S \) closed under the action of \( \{ \psi^k \} \), then \( n \nmid p-1 \).

This is analogous to the discussion in Steenrod [9], in which a proof of the necessity of \( n \nmid p-1 \) is sketched using the secondary cohomology operations of [8]. Some more general results on polynomial rings with \( \{ \psi^k \} \) actions are given in [11].

REFERENCES