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SPHERES WHICH ARE LOOP SPACES mod p

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ABSTRACT. If $S^{2n-1}_{(p)}$ has a loop space structure, than $n|p-1$.

Which spheres have H-structures or loop structures is known, see **[1]**. The H-space structure mod p version was answered by **[2]** and **[10]**; and recently, Sullivan **[10]** has shown the following for odd prime p:

THEOREM 1. *$S^{2n-1}_{(p)}$ has a loop space structure if and only if $n|p-1$.*

The purpose of this note is to present a proof of the necessity that $n|p-1$, via calculations with the Adams operations in K-theory, as opposed to the usual secondary cohomology operations **[9]**, **[8]**. For an account of the Adams operations, see **[3]** and **[4]**. For other applications of the Adams operations, see **[5]**, **[6]**, **[7]**, and **[11]**. Our notation will follow **[10]**. That is,

$$Z_{(p)} = \{\text{integers localized at } p\},$$
$$= \{\text{rationals with denominator prime to } p\};$$

$X_{(p)}$ is the p-localization of X, for X a simple CW complex.

DEFINITION 1. If K is a filtered algebra over $Z_{(p)}$, K has a $\{\psi^k\}$ action if and only if there exists a sequence of filtered algebra homomorphisms $\{\psi^k\}$ satisfying the properties of the Adams operations as listed in **[4]**. That is,

(i) $\psi^p x = x^p \bmod p$,

(ii) $\psi^p\psi^q = \psi^q\psi^p$,

(iii) $\psi^k x - k^n x \in K_{n+1}$, if $x \in K_n$,

(iv) if $x \in K_n$, $\exists v_{n+i} \in K_{n+i(p-1)}$ so that $\psi^p x = \sum_i p^{n-i} v_{n+i}$ where $0 \leqq i \leqq n$.

In particular, $K(X)\otimes Z_{(p)}$ has a $\{\psi^k\}$ action, for any finite CW complex X, with $K_n = \ker\{i: X_{2n-1} \to X\}^*$.

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THEOREM 2. *If $K=Z_{(p)}[x_n]/(x_n^{p+1})$ has a $\{\psi^k\}$ action, then $n|p-1$. Here K is a truncated polynomial algebra on one generator x_n of degree $2n$, and K is filtered by even degrees.*

Thus, if $S^{2n-1}_{(p)}=\Omega BS^{2n-1}_{(p)}$, then for an N-skeleton, $K((BS^{2n-1}_{(p)})^N)\otimes Z_{(p)}$ will be a truncated polynomial algebra on one generator **[11]**. If N is large enough, we can truncate at level $(p+1)$. This preserves the $\{\psi^k\}$ action; hence Theorem 2 implies $n|p-1$, and the necessary part of Theorem 1 is completed.

PROOF OF THEOREM 2.

Notation. $r(i,j)=1$ if $ij\equiv 0 \bmod p-1$, $=0$ otherwise.

LEMMA 1. $\sum_{j<p} r(m,j)=\mathrm{GCD}(m,p-1)=(m,p-1)$.

LEMMA 2. *Given α, a positive integer, there exists q, an integer, so that for all $\beta\leqq\alpha+1$, $q^k-1\equiv 0 \bmod p^\beta$ if and only if* (a) $p-1|k$ *and* (b) $p^{\beta-1}|k$.

PROOF. Choose q a generator of the units in $Z/p^{\alpha+1}$.

DEFINITION 2. If there is a $\{\psi^k\}$ action on $Z_{(p)}[x_n]/(x_n^{p+1})$, denote by $\langle\psi^k x, x^s\rangle$ the coefficient of x^s in $\psi^k x$.

LEMMA 3. $\langle\psi^p x, x^i\rangle=0 \bmod p^{n-(\sum_{j<i} r(m,j))(\alpha+1)}$ *where $n=mp^\alpha$ and $p\nmid m$ and $i\leqq p$. Denote this exponent by N_i.*

PROOF. The proof of Lemma 3 is by induction on i. For $i=1$, the statement of Lemma 3 is $\langle\psi^p x, x\rangle=0 \bmod p^n$. This is true by property (iii). Choose a q as in Lemma 2, and let $i<p$.

By our notation, $\psi^p x=\sum_{s\geq 1}\langle\psi^p x, x^s\rangle x^s$, $\psi^q x=\sum_{s\geq 1}\langle\psi^q x, x^s\rangle x^s$. Therefore,

$$\psi^p\psi^q x=\sum_{s\geqq 1}\langle\psi^q x, x^s\rangle(\psi^p x)^s=\sum_{s\geqq 1}\left(\sum_{k\geqq 1}\langle\psi^p x, x^k\rangle x^k\right)^s\langle\psi^q x, x^s\rangle.$$

Similarly,

$$\psi^q\psi^p x=\sum_{s\geqq 1}\left(\sum_{k\geqq 1}\langle\psi^q x, x^k\rangle x^k\right)^s\langle\psi^p x, x^s\rangle.$$

The inductive hypothesis is that $\langle\psi^p x, x^k\rangle=0 \bmod p^{N_k}$, for $k\leqq i<p$. In particular, $\langle\psi^p x, x^k\rangle=0 \bmod p^{N_i}$, for $k\leqq i<p$. Looking only at x^{i+1} in the above expansions we see that

$$\langle\psi^q\psi^p x, x^{i+1}\rangle=\langle\psi^p x, x^{i+1}\rangle(\langle\psi^q x\ \ x\rangle)^{i+1} \bmod p^{N_i},$$

$$\langle\psi^p\psi^q x, x^{i+1}\rangle=\langle\psi^p x, x^{i+1}\rangle\langle\psi^q x, x\rangle \bmod p^{N_i}.$$

But $\psi^q\psi^p-\psi^q\psi^p=0$ by (ii). So

$$\langle\psi^p x, x^{i+1}\rangle[\langle\psi^q x, x\rangle^{i+1}-\langle\psi^q x, x\rangle]=0 \bmod p^{N_i}.$$

Since $\langle\psi^q x, x\rangle = q^n$ by property (iii), we have $q^n(q^{in}-1)\langle\psi^p x, x^{i+1}\rangle = 0 \bmod p^{N_i}$. By Lemma 2, $q^{in}-1=0 \bmod p^{r(i,n)(\alpha+1)}$. But $r(i, n)=r(i, m)$, so $\langle\psi^p x, x^{i+1}\rangle=0 \bmod p^{N_i-r(i,m)(\alpha+1)}$. Since $N_{i+1}=N_i-r(i, m)(\alpha+1)$, we have established the lemma.

The proof of Theorem 2 is completed by observing that for $i=p$, Lemma 3 gives $\langle\psi^p x, x^p\rangle=0 \bmod p^{n-(m,p-1)(\alpha+1)}$, and $n-(m, p-1)(\alpha+1) \geqq 1$ if $\alpha>0$. Also, $m-(m, p-1) \geqq 1$ if $m \nmid p-1$. Since by property (i) of the Adams operations, $\langle\psi^p x, x^p\rangle \neq 0 \bmod p$, $\alpha=0$ and $m \mid p-1$.

REMARK. An alternate statement of these results can be given with a more general formulation:

THEOREM 3. *If Y is a* CW *space such that $K(Y)\otimes \boldsymbol{Z}_{(p)}=R\oplus S$ with $R \cong \boldsymbol{Z}_{(p)}[x]/(x^r)$ with $r>p$, $x\in K_n$, $x\notin K_{n+1}$ and S closed under the action of $\{\psi^k\}$, then $n \mid p-1$.*

This is analogous to the discussion in Steenrod **[9]**, in which a proof of the necessity of $n \mid p-1$ is sketched using the secondary cohomology operations of **[8]**. Some more general results on polynomial rings with $\{\psi^k\}$ actions are given in **[11]**.

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