LOCAL BOUNDEDNESS AND CONTINUITY FOR A FUNCTIONAL EQUATION ON TOPOLOGICAL SPACES

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Abstract. It is known that the locally bounded solutions \( f \) of Cauchy's functional equation \( f(x) + f(y) = f(x + y) \) on the reals are necessarily continuous. We shall extend this result to the functional equation \( f(x) + g(y) = h(T(x, y)) \) on topological spaces.

1. Introduction. Let \( X, Y \) be topological spaces and let \( f: X \to \mathbb{R} \) (the reals), \( g: Y \to \mathbb{R} \), \( T: X \times Y \to \mathbb{R} \) and \( h: T(X \times Y) \to \mathbb{R} \) be functions satisfying the functional equation

\[
(1) \quad f(x) + g(y) = h(T(x, y))
\]

for all \( x \in X, \ y \in Y \). We shall give some sufficient topological assumptions on \( X \) and \( T \) so that the local boundedness and nonconstancy of \( f \) insure that \( g \) is continuous. The method was suggested by the work of J. Pfanzagl in his paper [6] generalizing a result of G. Darboux [2].

2. Main theorems.

Theorem 1. For equation (1), if each pair of points of \( X \) is contained in the continuous image of some connected and locally connected space (for instance, when \( X \) is connected and locally connected or when \( X \) is pathwise connected), \( T \) is continuous in each of its two variables and \( f \) is nonconstant and locally bounded from above (or from below) at each point of \( X \), then \( g \) is continuous on \( Y \).

Proof. Let \( a, b \in X \) be such that \( f(a) \neq f(b) \). There exist a connected and locally connected space \( \tilde{X} \) and a continuous mapping \( \gamma: \tilde{X} \to X \) such that \( a, b \in \gamma(\tilde{X}) \). The functions \( \tilde{f} := f \circ \gamma \) and \( \tilde{T} \) with \( \tilde{T}(\tilde{x}, y) := T(\gamma(\tilde{x}), y) \) for \( \tilde{x} \in \tilde{X}, y \in Y \), now satisfy the induced functional equation

\[
(\tilde{1}) \quad \tilde{f}(\tilde{x}) + g(y) = h(\tilde{T}(\tilde{x}, y))
\]
for all $x \in X$, $y \in Y$. The local boundedness of $f$ passes to $\bar{f}$ and the continuity of $T$ in each variable passes to $\bar{T}$. With this observation there is no loss of generality if we suppose from the very beginning that $X$ is connected and locally connected.

Since $X$ is connected and $f$ is nonconstant on $X$, $f$ cannot be locally constant on $X$ and there exists a point $e \in X$ such that $f$ is nonconstant on every neighbourhood of $e$. As $X$ is locally connected and $f$ is locally bounded from above at $e$ there exists an open connected neighbourhood $U$ of $e$ on which $f$ is bounded from above. Thus $f$ is nonconstant and bounded from above on the connected and locally connected set $U$.

Let $x_1$, $x_2 \in U$ be such that $f(x_1) \neq f(x_2)$. It follows from equation (1) that $T(x_1, y) \neq T(x_2, y)$ for all $y \in Y$.

Let $y_0 \in Y$ be arbitrarily given and we shall prove the continuity of $g$ at $y_0$. We may suppose that $t_1 := T(x_1, y_0) < T(x_2, y_0) =: t_2$. By Lemma 1 in Pfanzagl [5] there exists a connected $B \subseteq U$ such that $T(B, y_0) = \{t_1, t_2\}$. Let $\epsilon > 0$ be arbitrarily given. Since $\sup f(B) < \infty$, there exists $x_0 \in B$ such that $f(x_0) \geq f(x) - \epsilon$ for all $x \in B$.

Let $M := \{y \in Y : t_1 < T(x_0, y) < t_2\}$. Then $y_0 \in M$ and, as $T(x_0, \cdot)$ is continuous on $Y$, $M$ is a neighbourhood of $y_0$.

For each $y \in M$, $T(x_0, y) \in [t_1, t_2[ = T(B, y_0)$ and so there exists $x \in B$ such that $T(x_0, y) = T(x, y_0)$. Thus $f(x_0) + g(y) = h(T(x_0, y)) = h(T(x, y_0)) = f(x) + g(y_0)$. As $f(x_0) \geq f(x) - \epsilon$ we have $g(y) \leq g(y_0) + \epsilon$.

Let $x_3, x_4 \in B$ be arbitrarily chosen such that $t_3 := T(x_3, y_0) < T(x_0, y_0) < t_4 := T(x_4, y_0)$.

Let $N := \{y \in Y : T(x_3, y) < T(x_0, y_0) < T(x_4, y)\}$. Then $y_0 \in N$ and, as $T(x_0, \cdot)$ and $T(x_1, \cdot)$ are continuous on $Y$, $N$ is a neighbourhood of $y_0$.

For each $y \in N$, $T(B, y)$ is an interval of $R$ as $B$ is connected and $T(\cdot, y)$ is continuous. Furthermore, $T(x_3, y)$ and $T(x_4, y)$ are points of $T(B, y)$ with $T(x_3, y) < T(x_0, y_0) < T(x_4, y)$ and so $T(x_0, y_0) \in T(B, y)$. Hence there exists $x \in B$ such that $T(x_0, y_0) = T(x, y)$. From this we have $f(x_0) + g(y_0) = f(x) + g(y)$. As $f(x_0) \geq f(x) - \epsilon$ we have $g(y_0) - \epsilon \leq g(y)$. $M \cap N$ is then a neighbourhood of $y_0$ and $g(y_0) - \epsilon \leq g(y) \leq g(y_0) + \epsilon$ for every $y \in M \cap N$. This proves the continuity of $g$ at $y_0$.

Remark 1. Lemma 1 in Pfanzagl [5] is given as: let $X$ be a connected and locally connected Hausdorff space, $\theta : X \to R$ a continuous map, then to any $t_1$, $t_2 \in \theta(X)$ with $t_1 < t_2$, there exists a connected component $B$ of $\theta^{-1}(\{t_1, t_2\})$ such that $\theta(B) = \{t_1, t_2\}$. The proof is based on a theorem of Wilder [7, p. 46, Theorem 3.8]. The assumption that $X$ is Hausdorff is however not used and can be removed.

Corollary 1. If $X$ is locally connected, $T : X \times X \to R$ is continuous in each variable, $f : X \to R$ is locally bounded from above (or from below) at
each point of $X$ and $h$ is any function on $T(X, X)$ satisfying the functional equation

$$f(x) + f(y) = h(T(x, y))$$

for all $x, y \in X$, then $f$ must be continuous on $X$.

**Proof.** For a point $a \in X$, if $f$ is locally constant at $a$ then $f$ is continuous at $a$. We may suppose now $f$ is not locally constant at $a$ and hence there exists an open connected neighbourhood $U$ of $a$ such that $f$ is bounded and nonconstant on $U$. We can apply Theorem 1 to the equation

$$f(x) + f(y) = h(T(x, y))$$

for all $x \in U$, $y \in X$ yielding the continuity of $f$ on $X$.

**Remark 2.** Corollary 1 is proved by Pfanzagl [6] under stronger assumptions on $X$—that $X$ is locally compact and locally connected Hausdorff.

**Theorem 2.** For equation (1), if each pair of points of $X$ is contained in some compact connected subset of $X$, $T$ is jointly continuous on the product space $X \times Y$ and $f$ is nonconstant and locally bounded from above on $X$ (or locally bounded from below on $X$), then $g$ is continuous on $Y$.

**Proof.** Similar to the argument given in the first paragraph in the proof of Theorem 1 we may suppose that $X$ is compact and connected. We note that $f$ is then bounded from above on every subset of $X$.

Let $y_0 \in Y$ and $\epsilon > 0$ be arbitrarily given.

Since $f$ is nonconstant on $X$, for each $y \in Y$ the function $T(\cdot, y)$ is nonconstant on $X$ and $T(X, y)$ is a proper closed interval of $R$. Write $T(X, y_0) = [a, b]$ with $a < b$. Let $A = \{x \in X: T(x, y_0) = a\}$, $B = \{x \in X: T(x, y_0) = b\}$ and $C = \{x \in X: a < T(x, y_0) < b\}$. The sets $A$, $B$ and $C$ partitioned $X$ with $A$ and $B$ being closed in $X$ and therefore compact. Since $\sup f(C) < \infty$ there exists $x_0 \in C$ such that $f(x_0) = \inf f(x) - \epsilon$ for all $x \in C$.

We first let $M = \{y \in Y: a < T(x_0, y) < b\}$.

Similar to the proof lines in Theorem 1, $M$ is seen to be a neighbourhood of $y_0$ and $g(y) \leq g(y_0) + \epsilon$ for all $y \in M$.

Secondly, we let $N = \{y \in Y: T(x, y) < T(x_0, y_0) < T(x', y) \text{ for all } x \in A, x' \in B\}$. We proceed to show that $N$ is a neighbourhood of $y_0$.

For each $x \in A$ we have $T(x, y_0) = a \in ]-\infty, T(x_0, y_0)[$. $T$ is jointly continuous and so there exist neighbourhoods $U(x)$, $V_x(y_0)$ of $x$ and $y_0$ respectively such that $T(U(x), V_x(y_0)) \subset ]-\infty, T(x_0, y_0)[$. Now, because $A$ is compact, there exists a finite subset $A' \subseteq A$ such that $\bigcup \{U(x): x \in A'\} \supseteq A$. The finite intersection $V' = \bigcap \{V_x(y_0): x \in A'\}$ is then a neighbourhood of $y_0$ and $T(A, V') \subset ]-\infty, T(x_0, y_0)[$. Similarly, there exists a
neighbourhood \( W \) of \( y_0 \) such that \( T(B, W) \subseteq T(x_0, y_0) \). Now \( N \supseteq V \cap W \) and is a neighbourhood of \( y_0 \).

For each \( y \in N \), \( T(X, y) \) is an interval of \( R \). The fact that \( T(A, y) \subseteq ]-\infty, T(x_0, y_0)\) and \( T(B, y) \subseteq T(x_0, y_0) \) implies \( T(x_0, y_0) \subseteq T(C, y) \). Thus there exists \( x \in C \) such that \( T(x_0, y_0) = T(x, y) \). It follows that \( f(x_0) + g(y_0) = f(x) + g(y) \). Since \( f(x_0) \geq f(x) - \varepsilon \) we have \( g(y_0) - \varepsilon \leq g(y) \).

Theorem 3. For equation (1), if \( X \) is connected, \( T \) is continuous in each variable and \( f \) is nonconstant and bounded on \( X \) (from both sides), then \( g \) is continuous on \( Y \).

Proof. Let \( y_0 \in Y \) and \( \varepsilon > 0 \) be arbitrarily given.

The nonconstancy of \( f \) in equation (1) implies that \( T(\cdot, y_0) \) is non-constant. \( T(X, y_0) \) is then a nondegenerated interval of \( R \) and there exist \( t_1, t_2 \in T(X, y_0) \) with \( t_1 < t_2 \). The set \( B = \{ x \in X : T(x, y_0) \in ]t_1, t_2[ \} \) is mapped by \( T(\cdot, y_0) \) onto \( ]t_1, t_2[ \). Since \( f \) is bounded from above on \( B \) there exists \( x_0 \in B \) such that \( f(x_0) \geq f(x) - \varepsilon \) for all \( x \in B \). If we set

\[
M := \{ y \in Y : T(x_0, y) \in ]t_1, t_2[ \}
\]

we see that \( M \) is a neighbourhood of \( y_0 \). Furthermore for each \( y \in M \), \( T(x_0, y) \in ]t_1, t_2[ = T(B, y_0) \) and so there exists \( x \in B \) such that \( T(x_0, y) = T(x, y_0) \). It follows that \( f(x_0) + g(y) = f(x) + g(y_0) \). As \( f(x_0) \geq f(x) - \varepsilon \) we have \( g(y) \leq g(y_0) + \varepsilon \).

The above argument applies to the functions \( f = -f, \tilde{g} = -g \), and \( h = -h \) satisfying again equation (1). Hence there exists a neighbourhood \( \tilde{M} \) of \( y_0 \) such that \( \tilde{g}(y) \leq g(y_0) + \varepsilon \) for all \( y \in \tilde{M} \), i.e. \( g(y_0) - \varepsilon \leq g(y) \).

On the neighbourhood \( M \cap \tilde{M} \) we have \( g(y_0) - \varepsilon \leq g(y) \leq g(y_0) + \varepsilon \) for every \( y \in M \cap \tilde{M} \). This proves the continuity of \( g \).

3. Some examples. The connectedness of \( X \) is a common assumption in Theorems 1, 2 and 3. Its essentiality can be seen from the following example.

Example 1. We take \( X = \{0, 1\} \) the discrete space \( \subseteq R, Y = R \) the reals with the usual topology, \( T(x+y) = x+y, f: X \to Y \) the natural inclusion map, \( g = h: R \to R \) an additive function of the reals which is continuous at no place and leaving the rationals fixed. Obviously equation (1) is satisfied, \( X \) is locally connected and compact, \( T \) is jointly continuous and \( f \) is bounded, nonconstant on \( X \), while \( g \) is continuous at no place.

However, connectedness of \( X \) alone is not sufficient to give Theorems 1 and 2. This has been shown by C. Hipp who gave the following example.
Example 2 (by C. Hipp). Let $X = Y = \mathbb{R}$, $Y$ endowed with the canonical topology $\tau$ on $\mathbb{R}$ and $X$ endowed with the topology $\tau_1$ generated by $\tau$ and all subsets of $\mathbb{R}$ containing the rational numbers $\mathbb{Q}$. Then $(X, \tau_1)$ is connected. Let $\phi$ be a discontinuous (with respect to $\tau$) solution of the Cauchy equation

$$\phi(x) + \phi(y) = \phi(x + y) \quad \text{for all } x, y \in \mathbb{R}.$$ 

As for each $x \in X$, $\phi$ is bounded on $\{(x) \cup \mathbb{Q}\} \cap (x-1, x+1)$ which is a $\tau_1$ neighbourhood of $x$, we have the local boundedness of $\phi$ on $(X, \tau_1)$. The map $T$ with $T(x, y) = x + y$ is jointly continuous on $(X, \tau) \times (Y, \tau)$ and hence continuous on $(X, \tau_1) \times (Y, \tau)$. However $\phi$ is continuous on $(Y, \tau)$ at no place.

The local connectedness of $X$ for Corollary 1 is by no means redundant. We illustrate this by the following example.

Example 3. We take $X = \{n^{-1}: n = 1, 2, \cdots\} \cup \{0\}$ as a subspace of $\mathbb{R}$, $T(x, y) = x + y/2$ on $X \times X$, $f(x) = 0$ if $x \neq 0$ and $f(0) = 1$, $h(n^{-1}) = h(n^{-1}/2) = 1$ for all $n = 1, 2, \cdots$ and $h(n^{-1} + m^{-1}/2) = 0$ for all $n, m = 1, 2, \cdots$ and $h(0) = 2$. Obviously, equation (2) is satisfied, $X$ fails to be locally connected at $0$, $T$ is jointly continuous, $f$ is bounded on $X$ but fails to be continuous at $0$.

Some uniqueness theorems concerning the continuous solutions of equations (1) and (2) are given in Ng [4] and Pfanzagl [5].

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