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# INTERSECTING UNIONS OF MAXIMAL CONVEX SETS

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ABSTRACT. Hare and Kenelly have characterized the intersection of the maximal starshaped subsets of a set $S$, where $S$ is compact, simply connected and planar, and Sparks has solved the general problem for maximal $L_n$ sets. In this paper, a similar question is examined for unions of maximal convex sets: Let $S$ be a subset of $R^2$, $\mathscr{C}$ the collection of all maximal convex subsets of $S$, and $\mathscr{N}=\{A\cup B: A, B \text{ distinct members of } \mathscr{C}\}$. Then $\bigcap \mathscr{N}$ is expressible as a union of three or fewer convex sets.

## 1. Intersecting unions of two maximal convex sets.

LEMMA 1. *Let $\mathscr{C}$ be any family of sets and $\mathscr{M}=\{A_1\cup\cdots\cup A_k: A_1,\cdots, A_k$ distinct members of $\mathscr{C}\}$. Then $x\in\bigcap\mathscr{M}$ if and only if there are at most $k-1$ members of $\mathscr{C}$ which fail to contain $x$.*

THEOREM 1. *Let $\mathscr{C}$ be any collection of closed convex subsets of the plane and let $\mathscr{M}=\{A\cup B: A, B$ distinct members of $\mathscr{C}\}$. Then $\bigcap\mathscr{M}$ can be expressed as a union of three or fewer closed convex sets.*

PROOF. We assume that $\bigcap\mathscr{M}$ is not convex and consists of more than three points, and that $\mathscr{C}$ has at least three distinct members, for otherwise the result is trivial. We examine two cases.

*Case* 1. Assume that $\bigcap\mathscr{M}$ is three convex. That is, for $x, y, z$ in $\bigcap\mathscr{M}$, at least one of the corresponding segments lies in $\bigcap\mathscr{M}$. Since $\bigcap\mathscr{M}$ is closed, if it is connected, then by a theorem of Valentine **[3]**, $\bigcap\mathscr{M}$ is expressible as a union of three or fewer closed convex sets having a nonempty intersection, completing the proof. If $\bigcap\mathscr{M}$ is not connected, then it has exactly two closed components, each of which is necessarily convex by the three convexity of $\bigcap\mathscr{M}$. This completes Case 1.

*Case* 2. If $\bigcap\mathscr{M}$ is not three convex, there are points $x, y, z$ in $\bigcap\mathscr{M}$ for which none of the corresponding segments lie in $\bigcap\mathscr{M}$. Thus there is some $A\cup B$ in $\mathscr{M}$ not containing all three segments. Assume $x, y\in A$, $z\in B\sim A$. Then $A$ is the only member of $\mathscr{C}$ not containing $z$ (by Lemma 1). Since $[x, y]\not\subseteq\bigcap\mathscr{M}$, there is some $C\cup D$ in $\mathscr{M}$ not containing $[x, y]$, and without loss of generality we may assume $x\in C\sim D$, $y\in D\sim C$, $z\in D$.

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Again by Lemma 1, $D$ is the only member of $\mathscr{C}$ not containing $x$, $C$ the only one not containing $y$. Also, since $y$, $z \in D$, $[y, z] \subseteq D$. Since $y \in A \sim C$, $A \neq C$, and $A \cup C$ belongs to $\mathscr{M}$. Thus $z \in C$ and $[x, z] \subseteq C$.

Moreover, for any member $E$ of $\mathscr{C}$ distinct from each of $A$, $C$, $D$, by Lemma 1, $E$ necessarily contains $x$, $y$, and $z$.

We examine the following closed convex subsets of $\bigcap \mathscr{M}$. Define $A_0 = \bigcap \{E : E \text{ in } \mathscr{C}, E \neq A\}$, $C_0 = \bigcap \{E : E \text{ in } \mathscr{C}, E \neq C\}$, $D_0 = \bigcap \{E : E \text{ in } \mathscr{C}, E \neq D\}$. We will show that the only points of $\bigcap \mathscr{M}$ which fail to be in $A_0 \cup C_0 \cup D_0$ necessarily lie in $A \cap C \cap D \subseteq \ker(\bigcap \mathscr{M})$. It will then be easy to express $\bigcap \mathscr{M}$ as a union of three closed convex sets.

We begin by showing that $A \cap C \cap D$ is in $\operatorname{conv}\{x, y, z\}$. Let $p \in A \cap C \cap D$. If $[y, z]$ contained $p$, then $[y, p] \subseteq A$, $[p, z] \subseteq C$, and $[y, z] \subseteq A \cup C$. However $[y, z] \subseteq E$ for $E \neq A$, $C$, $D$. Also $[y, z] \subseteq D$, so $[y, z]$ would lie in $\bigcap \mathscr{M}$, a contradiction since none of the segments determined by $x, y, z$ lie in $\bigcap \mathscr{M}$.

A parallel argument shows that neither $[x, y]$ nor $[x, z]$ contains a point of $A \cap C \cap D$. Now for $p$ in $A \cap C \cap D$, if $[p, x]$ cut $[y, z]$ at $q$, then $[p, x] \subseteq A \cap C$ and $q \in A \cap C \cap D$ which is impossible by the preceding paragraph. Similarly $[p, y]$ cannot cut $[x, z]$, $[p, z]$ cannot cut $[x, y]$.

Also, $x \notin \operatorname{conv}\{p, y, z\}$ (for otherwise $x$ would lie in $D$), $y \notin \operatorname{conv}\{p, x, z\}$, and $z \notin \operatorname{conv}\{p, x, y\}$.

Hence $p$ must be interior to $\operatorname{conv}\{x, y, z\}$, and since $x$, $y$, $z \in E$ for every $E \neq A$, $C$, $D$, it follows that $p$ is in every member of $\mathscr{C}$ and in $\bigcap \mathscr{M}$.

Moreover, $p \in \ker(\bigcap \mathscr{M})$, for if $t \in \bigcap \mathscr{M}$, $t$ fails to belong to at most one $E$ in $\mathscr{C}$, so $[p, t]$ fails to lie in at most one member of $\mathscr{C}$, and $[p, t] \subseteq \bigcap \mathscr{M}$.

Now examine the sets $A_0$, $C_0$, $D_0$ defined previously. It is clear that each of these sets lies in $\bigcap \mathscr{M}$ by Lemma 1. For $u$ in $\bigcap \mathscr{M}$, either $u$ fails to lie in one of $A$, $C$, or $D$ (and hence lies in one of $A_0$, $C_0$, or $D_0$), or $u$ lies in $A \cap C \cap D$. Since $A \cap C \cap D \subseteq \ker(\bigcap \mathscr{M})$, the set

$$\operatorname{conv}((A \cap C \cap D) \cup A_0)$$

is a subset of $\bigcap \mathscr{M}$. Thus each of the sets $A_1 = \operatorname{cl} \operatorname{conv}((A \cap C \cap D) \cup A_0)$, $C_1 = \operatorname{cl} \operatorname{conv}((A \cap C \cap D) \cup C_0)$, $D_1 = \operatorname{cl} \operatorname{conv}((A \cap C \cap D) \cup D_0)$ is a closed convex subset of the closed set $\bigcap \mathscr{M}$, and $\bigcap \mathscr{M} = A_1 \cup C_1 \cup D_1$, completing the proof.

Remark. It is easy to find examples which show that the number three in Theorem 1 is best possible. (See Example 1 of this paper.)

Using Theorem 1, it is possible to prove the following.

Theorem 2. *Let $S$ be planar, $\mathscr{C}$ the collection of all maximal convex subsets of $S$. Let $\mathscr{N} = \{A \cup B : A, B \text{ distinct members of } \mathscr{C}\}$. Then $\bigcap \mathscr{N}$ can be expressed as a union of three or fewer convex sets.*

PROOF. By an easy application of Theorem 1, for $\mathscr{M}=\{\text{cl } A\cup\text{cl } B: A, B$ distinct members of $\mathscr{C}\}$, $\bigcap \mathscr{M}$ is a union of three or fewer closed convex sets $S_i$, $i=1, 2, 3$.

Let $M_i=S_i\cap(\bigcap \mathscr{N})$, $i=1, 2, 3$. If each $M_i$ is convex, the proof is complete. Assume otherwise to reach a contradiction. Suppose for $v$, $w$ in $M_1$, $[v, w]\nsubseteq M_1$. Then for some $p$, $v<p<w$, $p\notin M_1$. Therefore, there exist sets $G$, $F$ in $\mathscr{C}$ with $p\notin G\cup F$. Without loss of generality, assume $v\in G\sim F$, $w\in F\sim G$.

If $G$, $F$ are the only members of $\mathscr{C}$, the proof is trivial. Otherwise, for every $E$ in $\mathscr{C}\sim\{G, F\}$, $[v, w]\subseteq E$ by Lemma 1. Thus $p\in[v, w]\subseteq E\subseteq S$. Also, since $p\in S_1$, $p\in \text{cl } G\cup\text{cl } F$, so assume $p\in\text{cl } G$. For every $x$ in $G$, if $[p, x]\subseteq S$, then the cone $pG\equiv\bigcup\{[p, x]: x \text{ in } G\}$ would be a convex subset of $S$ containing $G$. But since $G$ is maximal, this would imply that $p\in G$, a contradiction. Hence for some $x$ in $G$, $[x, p]\nsubseteq S$. Clearly such an $x$ cannot lie on the line $L(v, w)$ determined by $v$ and $w$, since $[v, w]\subseteq S$ and $[x, v]\subseteq G\subseteq S$.

For some $y$, $x<y<p$, $y\notin S$. Since $x, p\in\text{cl } G$, $y\in\text{cl } G$, and since $G$ is convex, $y$ must lie on bdry $G$. There is a supporting hyperplane $H$ to cl $G$ at $y$, and $H$ contains $[x, p]$ since $[y, p]\subseteq\text{cl } G\sim G\subseteq\text{bdry } G$. Note that this implies $x\in\text{bdry } G$, and therefore $p$ sees via $S$ all points interior to $G$. Clearly int $G\neq\varnothing$ since $x\notin L(v, w)$.

Consider the cone $G_1=p(\text{int } G)\equiv\bigcup\{[p, x]: x\in\text{int } G\}$. This is a convex subset of $S$. If necessary, extend $G_1$ to a maximal convex subset $G_2$ of $S$. It is easy to see that $w\notin G_2$: Let $U$ be any spherical neighborhood of $x$ disjoint from the line $L(w, y)$. Certainly $U$ contains points of int $G$, and for $x_1$ in $U\cap\text{int } G$, $y\in\text{conv}\{x_1, p, w\}$. If $w$ were in $G_2$, then $y\in G_2\subseteq S$, a contradiction since $y\notin S$.

Now $p\in G_2\sim G$, so $G\neq G_2$ and $G\cup G_2$ is in $\mathscr{N}$. Since $w\in M_1\subseteq\bigcap\mathscr{N}$, $w$ must lie in $G\cup G_2$, but this is clearly impossible by the preceding paragraph. Hence our assumption is false, each $M_i$ is convex, and $\bigcap \mathscr{N}$ is a union of three or fewer convex sets.

2. **The general case.** It would be interesting to obtain analogues of Theorems 1 and 2 for unions of $k$ convex sets. The following results, although for special cases, invite the conjecture that the appropriate bound is $k(k+1)/2$.

THEOREM 3. *Let $\mathscr{C}$ be any collection of $k+1$ closed convex subsets of the plane and let $\mathscr{M}=\{A_1\cup\cdots\cup A_k: A_1, \cdots, A_k$ distinct members of $\mathscr{C}\}$. Then $\bigcap\mathscr{M}$ is expressible as a union of $k(k+1)/2$ or fewer closed convex sets. The result is best possible for all $k$.*

PROOF. The proof is by induction. The result is trivial for $k=1$, and for $k=2$, the result is an immediate consequence of Theorem 1. Assume the theorem true for $2<k-1$ to prove for arbitrary $k$.

Select any set $A$ in $\mathscr{C}$ and define subsets $P$, $Q$ of $\bigcap \mathscr{M}$ in the following manner. Let

$P = \{x : x \in A$ and $x$ fails to lie in exactly $k - 1$ members of $\mathscr{C} \sim \{A\}\}$,

$Q = \{x : x$ fails to lie in at most $k - 2$ members of $\mathscr{C} \sim \{A\}\}$.

Note that $x \in Q$ if and only if either $x \in A$ and $x$ fails to lie in no more than $k-2$ members of $\mathscr{C}$ or $x \notin A$ and $x$ fails to lie in no more than $k-1$ members of $\mathscr{C}$. Using Lemma 1, it is clear that $P \cup Q = \bigcap \mathscr{M}$.

Examine the set $Q$. Now $\mathscr{C} \sim \{A\}$ is a collection of $k$ closed convex sets in the plane. Letting

$$\mathscr{N} = \{B_1 \cup \cdots \cup B_{k-1} : B_1, \cdots, B_{k-1} \text{ distinct members of } \mathscr{C} \sim \{A\}\},$$

by our induction hypothesis, $Q = \bigcap \mathscr{N}$ is expressible as a union of $(k-1)k/2$ or fewer closed convex sets.

Furthermore, any point of $P$ necessarily lies in exactly two members of $\mathscr{C}$, one of which is $A$. Letting $E_i = A \cap A_i$, $A_i$ in $\mathscr{C} \sim \{A\}$, $1 \leqq i \leqq k$, then $P = \bigcup_{i=1}^{k} E_i$.

Hence $P \cup Q = \bigcap \mathscr{M}$ is a union of $(k-1)k/2 + k = k(k+1)/2$ or fewer closed convex sets, completing the proof.

EXAMPLE 1. To see that the result in Theorem 3 is best possible, let $\mathscr{C}$ denote a collection of $k+1$ lines $L_i$, $1 \leqq i \leqq k+1$, every two intersecting and no three having a common point. Then the corresponding $\bigcap \mathscr{M}$ consists of exactly $k(k+1)/2$ isolated points.

In conclusion, we note that Example 1 reveals the "worst" case when $\mathscr{C}$ is any family of lines, for a proof paralleling that of Theorem 3 shows that the bound is again $k(k+1)/2$. The only additional step involves showing that for $A$ in $\mathscr{C}$, the corresponding $P$ may be represented as a union of $k$ or fewer convex sets: If more than $k$ convex sets were required, there would be at least $k+1$ distinct members of $\mathscr{C} \sim \{A\}$, each intersecting $A$ at a different point, and for $x$ in $A \cap (\bigcap \mathscr{M})$, $x$ would fail to lie in at least $k$ members of $\mathscr{C}$, contradicting Lemma 1. Thus $P$ has the desired representation and the result follows.

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