Palais established the following generalization of the Morse lemma:

**Theorem (Morse-Palais Lemma [1]).** Let $V$ be a real Banach space, $\emptyset$ a convex neighborhood of the origin and let $f : \emptyset \to \mathbb{R}$ be a $C^{k+2}$ function ($k \geq 1$) having the origin as a nondegenerate critical point with $f(0) = 0$. Then there is a neighborhood $U$ of the origin and a $C^k$ diffeomorphism $\Phi : U \to \emptyset$ with $\Phi(0) = 0$ and $(D\Phi)_0 = \text{id}_V$ (the identity map of $V$) such that for $x$ in $U$, $f(\Phi(x)) = \frac{1}{2}(D^2f)_0(x, x)$.

Here and henceforth, the nonexplained notations are standard and are the same as in [1]. Palais' proof of the theorem draws on the theory of differential equations. We shall give of Palais' theorem a very elementary and short proof. Our proof, which only uses a direct application of the inverse mapping theorem, is even shorter and more elementary than Palais' proof of his theorem for the Hilbert space case [2]. The idea of our proof came from a close examination of Palais' proof for the Hilbert space case (loc. cit.)

We first observe that since 0 is a nondegenerate critical point of $f$, $(D^2f)_0$ is an isomorphism of $V$ onto $V^*$ and $(Df)_0 = 0$.

**Remark 1.** The problem is obviously equivalent to that of establishing the existence of a $C^k$-isomorphism $\Psi$ of an open neighborhood $U_1$ of 0 contained in $\emptyset$ into $\emptyset$, such that $\Psi(0) = 0$, $(D\Psi)_0 = \text{id}_V$ and

$$f(y) = \frac{1}{2}(D^2f)_0(\Psi(y), \Psi(y)), \quad y \text{ in } U_1.$$  

For further use, we put $B = \frac{1}{2}(D^2f)_0$ and

$$E = \{h : h \in L(V, V) \text{ and } B(h(x))y = B(h(y))x, \text{ } x \text{ and } y \text{ in } V\}.$$  

Our proof of the Morse-Palais lemma depends on the following two simple lemmas.
**Lemma 1.**  $E$ is a closed subspace of $L(V, V)$ and the map

$$T : E \rightarrow L(V, V^*)$$

defined by $Th = B \circ h$ is an isomorphism of $E$ onto $L_s(V, V^*)$.

**Proof.** Let $Sh = B \circ h$, $h$ in $L(V, V)$. Then $S$ is an isomorphism of $L(V, V)$ onto $L(V, V^*)$ since $B$ is an isomorphism. Clearly, $T$ is the restriction of $S$ to $E$. It remains to show that $S$ maps $E$ onto $L_s(V, V^*)$.

It is clear from the definition of $E$ that $S$ maps $E$ into $L_s(V, V^*)$. Now let $C$ be in $L_s(V, V^*)$ and let $Sh = C$, i.e., $B \circ h = C$. Then

$$B(h(x))y = C(x)y = C(y)x = B(h(y))x.$$  

Hence $h$ is in $E$.  Q.E.D.

**Lemma 2.** Let $\theta : E \rightarrow L_s(V, V^*)$ be the map defined by

$$(\theta h)(x)y = B(h(x))h(y).$$

Then $\theta$ is a $C^\infty$-isomorphism of a neighborhood of $id_V$ onto a neighborhood of $B$ and $\theta(id_v) = B$.

**Proof.** By direct computation

$$[\theta(h + k) - \theta(h)](x)y = B(h(x))k(y) + B(k(x))h(y) + B(k(x))(k(y)).$$

Hence

$$(D\theta)_h(k)(x)y = B(h(x))k(y) + B(k(x))h(y).$$

Thus $D\theta$ is linear, and hence $\theta$ is $C^\infty$. Furthermore

$$(D\theta)_{id_v} k = 2B \circ k = 2Tk, \quad k \in E.$$  

By Lemma 1, $(D\theta)_{id_v}$ is therefore an isomorphism, and hence, Lemma 2 follows from the inverse mapping theorem.  Q.E.D.

**Proof of the Morse-Palais Lemma.** By repeated applications of the fundamental theorem of calculus, and using the fact that $f(0) = 0$, $(Df)_0 = 0$, we have

$$f(y) = \int_0^1 \int_0^1 (D^2f)_{xu} ds \, dt(y)y, \quad y \in \mathcal{O}.$$  

Put

$$G(y) = \int_0^1 \int_0^1 (D^2f)_{xu} ds \, dt.$$  

Then $G$ is a $C^k$-map of $\mathcal{O}$ into $L_s(V, V^*)$ and $G(0) = \frac{1}{2}(D^2f)_0 = B$.

By Lemma 2, there exists an open neighborhood $W$ of $B$ which is taken by the $C^\infty$-map $\theta^{-1}$ onto an open neighborhood of $id_V$. Now $G$ is in particular continuous, and hence maps an open neighborhood $U'$ of $0$
contained in \( \emptyset \) into \( W \). Put
\[
A(y) = \theta^{-1}(G(y)), \quad \Psi'(y) = A(y)y \quad \text{for } y \text{ in } U'.
\]
Note that \( A(y) \) is in \( L(V, V) \). It is readily seen that \( A \) and \( \Psi' \) are \( C^k \) on \( U' \), and \( A(0) = \text{id}_V \) and \( \Psi'(0) = 0 \).

Now
\[
(D\Psi)'_y = A(y) + (DA)'_y, \quad (D\Psi)'_0 = A(0) = \text{id}_V.
\]
Hence, by the inverse mapping theorem, \( \Psi' \) maps an open neighborhood \( U_1 \) of \( 0 \) contained in \( U' \) (hence contained in \( \emptyset \)) onto an open neighborhood \( U \) of \( 0 \) contained in \( \emptyset \). Furthermore
\[
f(y) = G(y)(y)y = \theta(A(y))(y)y
= B(A(y))y(A(y))y = B(\Psi'(y), \Psi'(y)), \quad y \text{ in } U_1.
\]
The theorem follows now from Remark 1 above.

REFERENCES


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