AN ELEMENTARY PROOF OF THE
MORSE-PALAIS LEMMA FOR BANACH SPACES

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Palais established the following generalization of the Morse lemma:

**Theorem (Morse-Palais Lemma [1]).** Let $V$ be a real Banach space, $\mathcal{O}$ a convex neighborhood of the origin and let $f: \mathcal{O} \to \mathbb{R}$ be a $C^{k+2}$ function ($k \geq 1$) having the origin as a nondegenerate critical point with $f(0) = 0$. Then there is a neighborhood $U$ of the origin and a $C^k$ diffeomorphism $\Phi: U \to \mathcal{O}$ with $\Phi(0) = 0$ and $(D\Phi)_0 = \text{id}_V$ (the identity map of $V$) such that for $x$ in $U$, $f(\Phi(x)) = \frac{1}{2}(D^2f)_0(x, x)$.

Here and henceforth, the nonexplained notations are standard and are the same as in [1]. Palais' proof of the theorem draws on the theory of differential equations. We shall give of Palais' theorem a very elementary and short proof. Our proof, which only uses a direct application of the inverse mapping theorem, is even shorter and more elementary than Palais' proof of his theorem for the Hilbert space case [2]. The idea of our proof came from a close examination of Palais' proof for the Hilbert space case (loc. cit.)

We first observe that since $0$ is a nondegenerate critical point of $f$, $(D^2f)_0$ is an isomorphism of $V$ onto $V^*$ and $(Df)_0 = 0$.

**Remark 1.** The problem is obviously equivalent to that of establishing the existence of a $C^k$-isomorphism $\Psi$ of an open neighborhood $U_1$ of $0$ contained in $\mathcal{O}$ into $\mathcal{O}$, such that $\Psi(0) = 0$, $(D\Psi)_0 = \text{id}_V$ and

$$f(y) = \frac{1}{2}(D^2f)_0(\Psi(y), \Psi(y)), \quad y \text{ in } U_1.$$

For further use, we put $B = \frac{1}{2}(D^2f)_0$ and

$$E = \{ h: h \in L(V, V) \text{ and } B(h(x))y = B(h(y))x, \text{ x and y in } V \}.$$

Our proof of the Morse-Palais lemma depends on the following two simple lemmas.

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Lemma 1. \( E \) is a closed subspace of \( L(V, V) \) and the map 
\[
T: E \to L(V, V^*)
\]
defined by \( Th = B \circ h \) is an isomorphism of \( E \) onto \( L_6(V, V^*) \).

Proof. Let \( Sh = B \circ h, h \in L(V, V) \). Then \( S \) is an isomorphism of \( L(V, V) \) onto \( L(V, V^*) \) since \( B \) is an isomorphism. Clearly, \( T \) is the restriction of \( S \) to \( E \). It remains to show that \( S \) maps \( E \) onto \( L_6(V, V^*) \).

It is clear from the definition of \( E \) that \( S \) maps \( E \) into \( L_6(V, V^*) \). Now let \( C \) be in \( L_6(V, V^*) \) and let \( Sh = C \), i.e., \( B \circ h = C \). Then
\[
B(h(x))y = C(x)y = C(y)x = B(h(y))x.
\]
Hence \( h \) is in \( E \). Q.E.D.

Lemma 2. Let \( \theta: E \to L_6(V, V^*) \) be the map defined by
\[
(\theta h)(x)y = B(h(x))h(y).
\]
Then \( \theta \) is a \( C^\infty \)-isomorphism of a neighborhood of \( \text{id}_V \) onto a neighborhood of \( B \) and \( \theta(\text{id}_V) = B \).

Proof. By direct computation
\[
[\theta(h + k) - \theta(h)](x)y = B(h(x))k(y) + B(k(x))h(y) + B(k(x))(k(y)).
\]
Hence
\[
(D\theta)_h(k)(x)y = B(h(x))k(y) + B(k(x))h(y).
\]
Thus \( D\theta \) is linear, and hence \( \theta \) is \( C^\infty \). Furthermore
\[
(D\theta)_{\text{id}_V} k = 2B \circ k = 2Tk, \quad k \in E.
\]
By Lemma 1, \( (D\theta)_{\text{id}_V} \) is therefore an isomorphism, and hence, Lemma 2 follows from the inverse mapping theorem. Q.E.D.

Proof of the Morse-Palais Lemma. By repeated applications of the fundamental theorem of calculus, and using the fact that \( f(0) = 0 \), \( (Df)_0 = 0 \), we have
\[
f(y) = \int_0^1 \int_0^1 (D^2f)_{ts}t \, ds \, dt \, y, \quad y \in \varnothing.
\]
Put
\[
G(y) = \int_0^1 \int_0^1 (D^2f)_{ts}t \, ds \, dt.
\]
Then \( G \) is a \( C^k \)-map of \( \varnothing \) into \( L_6(V, V^*) \) and \( G(0) = \frac{1}{2}(D^2f)_0 = B \).

By Lemma 2, there exists an open neighborhood \( W \) of \( B \) which is taken by the \( C^\infty \)-map \( \theta^{-1} \) onto an open neighborhood of \( \text{id}_V \). Now \( G \) is in particular continuous, and hence maps an open neighborhood \( U' \) of \( 0 \).
contained in $\mathcal{O}$ into $W$. Put
\[ A(y) = \theta^{-1}(G(y)), \quad \Psi(y) = A(y)y \quad \text{for } y \text{ in } U'. \]
Note that $A(y)$ is in $L(V, V)$. It is readily seen that $A$ and $\Psi$ are $C^k$ on $U'$, and $A(0) = \text{id}_V$ and $\Psi(0) = 0$.

Now
\[ (D\Psi)_y = A(y) + (DA)_y, \quad (D\Psi)_0 = A(0) = \text{id}_V. \]

Hence, by the inverse mapping theorem, $\Psi$ maps an open neighborhood $U_1$ of $0$ contained in $U'$ (hence contained in $\mathcal{O}$) onto an open neighborhood $U$ of $0$ contained in $\mathcal{O}$. Furthermore
\[
\begin{align*}
\theta(y) &= G(y)(y)y = \theta(A(y))(y)y \\
&= B(A(y))y(A(y))y = B(\Psi(y), \Psi(y)), \quad y \text{ in } U_1.
\end{align*}
\]

The theorem follows now from Remark 1 above.

REFERENCES


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