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STRUCTURE OF SEMIPRIME P.I. RINGS

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ABSTRACT. In this paper we make an investigation into the structure of semiprime polynomial identity rings which is culminated by showing that each such ring R has a unique maximal left quotient ring Q such that (1) Q is von Neumann regular with unity and (2) every regular element in R is invertible in Q.

Throughout this paper, R will denote an associative ring which does not necessarily have a unity. The *centroid* Ω of R is the ring of those endomorphisms ω of the additive group of R which have the property that $\omega(xy)=(\omega x)y=x(\omega y)$ for each x, y in R. Let $\{x_i\}$ be a set of noncommuting indeterminants over R and consider a polynomial $p(x)=p(x_1, x_2, \cdots, x_n)=\sum_\sigma \omega_\sigma x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$ which has coefficients $\omega_\sigma \in \Omega$ and σ varies over the symmetric group, S_n, of degree n. The *degree* of $p(x_1, x_2, \cdots, x_n)$ is n. We say that R *satisfies a polynomial identity* (*P.I.*) or simply that R is a *P.I. ring* if there exists a polynomial $p(x)$ such that (1) $p(r_1, r_2, \cdots, r_n)=0$ for all $r_1, r_2, \cdots, r_n \in R$ and (2) the kernel of ω_1 is zero.

Let T be a subring of R and let $S\subseteq T$. Then $\ell_T(S)=\{t\in T: tS=0\}$ ($\imath_T(S)=\{t\in T: St=0\}$). A left (right) ideal E of R is *essential* if each nonzero left (right) ideal of R has nonzero intersection with E. The *left* (*right*) *singular ideal* of R denoted $Z({}_RR)$ ($Z(R_R)$) is $\{a\in R: \ell(a)$ is essential$\}$ ($\{a\in R: \imath(a)$ is essential$\}$). Finally R is *semiprime* if R has no nonzero nilpotent ideals.

THEOREM 1. *Let R be a semiprime ring and let E be an essential left ideal of R which satisfies a polynomial identity. Then the left singular ideal of R is zero.*

PROOF. Assume that E satisfies

$$p(x) = p(x_1, x_2, \cdots, x_n) = \sum_\sigma \omega_\sigma x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}.$$

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We proceed by induction on n. When $n=1$, then the result follows since $E=0$ and hence $R=0$. Now assume that the result is true whenever the degree of the identity is less than n.

Suppose that $Z=Z({}_RR)\neq 0$. Then there exists $0\neq a\in Z$ and $Za\neq 0$ since R is semiprime. Moreover $T=[Za\cap\ell(a)\cap E]\neq 0$ and it can easily be verified that T satisfies $p(x)$. It follows from the semiprimeness of R that $\bar{T}=T/r_T(T)$ is a nonzero semiprime ring. If $x_1=a$ and $x_2, x_3, \cdots, x_n$ are arbitrary elements of T, then

$$0 = p(a, x_2, x_3, \cdots, x_n) = a\left(\sum_{\sigma} \omega_\sigma x_{\sigma(2)}x_{\sigma(3)}\cdots x_{\sigma(n)}\right)$$

since $T\subseteq\ell(a)$. Hence $p(x_2, x_3, \cdots, x_n)=\sum_\sigma \omega_\sigma x_{\sigma(2)}x_{\sigma(3)}\cdots x_{\sigma(n)}\subseteq r_T(T)$ where σ varies over S_{n-1}. Since $\omega_\sigma r_T(T)\subseteq r_T(T)$, we notice that each ω_σ induces an element in the centroid of $\bar{T}$ and the kernel of the element induced by ω_1 is zero. Thence $\bar{T}$ satisfies a polynomial identity of degree $n-1$. Wherefore, by the induction hypothesis $Z({}_T\bar{T})=0$.

We claim that $Z({}_T\bar{T})=\bar{T}$. Let $0\neq\overline{za}\in\bar{T}$, $z\in Z$ and let $\bar{L}\neq 0$ be a left ideal of $\bar{T}$. If $\overline{Lza}=0$, then $\bar{L}\cap\ell_T(\overline{za})\neq 0$. If $\overline{Lza}\neq 0$, then $TLza\neq 0$ and TL is a nonzero left ideal of R. Since $z\in Z$, we have $TL\cap\ell(z)\neq 0$ and so $T[TL\cap\ell(z)]\neq 0$. This yields $b\in L$ such that $Tb\neq 0$ with $bz=0$ which in turn yields $0\neq\bar{b}\in\bar{L}$ with $\overline{bza}=0$. Thus $\bar{L}\cap\ell_T(\overline{za})\neq 0$ and so $Z({}_T\bar{T})=\bar{T}\neq 0$. This contradiction forces $Z({}_RR)=0$ and completes the proof.

Corollary 2. *If R is a semiprime P.I. ring, then $Z({}_RR)=Z(R_R)=0$.*

Proof. Evident.

Theorem 3. *If R is a semiprime P.I. ring, then R has a unique maximal left quotient ring Q such that* (1) *Q is von Neumann regular with unity and* (2) *every regular element in R is invertible in Q.*

Proof. As $Z({}_RR)=0$ by Corollary 2, R has a unique maximal left quotient ring Q which is von Neumann regular with unity, ${}_RR\subseteq{}_RQ$ is an essential extension, and $Z({}_QQ)=0$ [**2**]. If a is a regular element in R, then $\ell_R(a)=0$ implies $\ell_Q(a)=0$. Since Q is von Neumann regular there exists b in Q such that $a=aba$. Hence $(1-ab)\in\ell_Q(a)=0$ or $ab=1$. Likewise, $\ell_Q(1-ba)\supseteq Ra$ which is essential in R by Herstein and Small [**1**, Theorem 3]. Thus $(1-ba)\in Z({}_QQ)=0$ or $ba=1$.

We conclude this paper by posing the following two questions: (1) Does the maximal left quotient ring Q which is obtained in Theorem 3 satisfy the same polynomial identity as R? and (2) For a semiprime P.I. ring R does the maximal right quotient ring, whose existence is guaranteed by Corollary 2, coincide with the maximal left quotient ring Q?

References

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