

SEMIGROUP RINGS AND SEMILATTICE SUMS OF RINGS

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ABSTRACT. A generalization of the concept of a decomposition of a ring into a direct sum of ideals is introduced. The question of semisimplicity of the ring in terms of the semisimplicity of its summands is investigated. The results are applied to semigroup rings.

1. Introduction. Many substantial results in the theory of rings either concern themselves with or use a decomposition of the ring into a direct sum of ideals. In this paper we introduce a generalization of direct sum decompositions. A ring is a supplementary semilattice sum of subrings if as an additive abelian group it is a direct sum of the additive groups of the subrings and if the subrings multiply in a certain natural way. We begin the investigation of semilattice sum decompositions by showing that in certain cases the semisimplicity of the subrings implies the semisimplicity of the ring. The investigation is carried out in the context of π -semisimplicity where π is a hereditary, homomorphic invariant property of rings and thus the result holds for Jacobson semisimplicity, nil semisimplicity and nilpotent semisimplicity (semiprime). The results are then applied to semigroup rings. We prove that if R is a commutative ring with identity and D is a commutative semigroup such that a power of each element lies in a subgroup then the semigroup ring RD is semiprime if and only if D is a semilattice of groups G_α , $\alpha \in \Omega$, and RG_α is semiprime for every $\alpha \in \Omega$. This generalizes Theorem 5.21 of Clifford and Preston [1]. In the paper subsequent to this one Janeski and the author study regularity of semilattice sum decompositions.

2. Preliminaries. Some concepts from the theory of semigroups may be useful to the reader. A *band* is a semigroup in which every element is idempotent ($x^2=x$). A *semilattice* is a commutative band. If Ω is a semilattice, the natural partial order is defined on Ω by $e \leq f$ if and only if $ef=fe=e$.

Received by the editors September 6, 1972.

AMS (MOS) subject classifications (1970). Primary 16A21, 16A48, 16A12, 20M25.

¹ This research was supported in part by U.S. Air Force Office of Scientific Research Grant AFOSR-72-2164.

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If R is a ring and R_α , $\alpha \in \Omega$, a collection of subrings then $\sum_{\alpha \in \Omega} R_\alpha = \{r_{\alpha_1} + \cdots + r_{\alpha_n} \mid r_{\alpha_i} \in R_{\alpha_i}\}$.

DEFINITIONS. A ring R is a *semilattice sum of subrings* R_α , $\alpha \in \Omega$, if Ω is a semilattice, $R = \sum_{\alpha \in \Omega} R_\alpha$ and $R_\alpha R_\beta \subseteq R_{\alpha\beta}$.

A ring R is a *supplementary semilattice sum of subrings* R_α , $\alpha \in \Omega$, if R is a semilattice sum of subrings R_α , $\alpha \in \Omega$, and if for every $\alpha \in \Omega$, $R_\alpha \cap \sum_{\beta \neq \alpha} R_\beta = \{0\}$; i.e., if the sum is direct considered as abelian groups under addition.

We shall often use the abbreviated expression, $R = \sum_{\alpha \in \Omega} R_\alpha$ is a (supplementary) semilattice sum.

REMARK. The concept of supplementary sum is mentioned in Divinsky [2]. Obviously the concepts of semilattice sum and supplementary semilattice sum can be generalized to band sum and supplementary band sum by removing the condition that Ω be commutative. If R is a direct sum of ideals I_α , $\alpha \in \Omega$, then it is a supplementary semilattice sum of the subrings I_α , $\alpha \in \Omega$, for if $\alpha, \beta \in \Omega$ then $I_\alpha I_\beta = \{0\}$ and therefore multiplication on Ω can be defined by $\alpha\beta = \alpha_0$ where α_0 is a fixed element of Ω .

If $R = \sum_{\alpha \in \Omega} R_\alpha$ is a supplementary semilattice sum and $x = x_{\alpha_1} + \cdots + x_{\alpha_n} \in R$ with $x_{\alpha_i} \in R_{\alpha_i}$, then we write $x(\alpha_i) = x_{\alpha_i}$. This defines a function $x: \Omega \rightarrow R$. It is well defined since in a supplementary sum the representation of each element is unique. Using this notation we have that if $x, y \in R$ and $\gamma \in \Omega$ then $xy(\gamma) = \sum_{\alpha\beta=\gamma} x(\alpha)y(\beta)$.

Continuing with the assumption that $R = \sum R_\alpha$ is a supplementary semilattice sum, we define, for $x \in R$, $\text{Supp}(x) = \{\alpha \in \Omega \mid x(\alpha) \neq 0\}$ and $\Gamma(x) = \langle \text{Supp}(x) \rangle$ the semigroup generated by $\text{Supp}(x)$. $\Gamma(x)$ is called the support semigroup of x and is a finite subsemigroup of Ω . Also for $\gamma \in \Omega$, define

$$R^\gamma = \sum_{\alpha \in \Omega: \alpha \leq \gamma} R_\alpha.$$

LEMMA 1. If $R = \sum_{\alpha \in \Omega} R_\alpha$ is a supplementary semilattice sum then for every $\gamma \in \Omega$, R^γ is an ideal of R and R_γ is a homomorphic image of R^γ .

PROOF. If $\alpha \leq \gamma$ then $\alpha\beta \leq \gamma$ for every $\beta \in \Omega$. Hence $R_\alpha R_\beta \subseteq R_{\alpha\beta} \subseteq R^\gamma$. So R^γ is an ideal.

For $x \in R^\gamma$ define $\varphi(x) = x(\gamma)$. Then $\varphi: R^\gamma \rightarrow R_\gamma$. Clearly φ is onto and $\varphi(x+y) = \varphi(x) + \varphi(y)$ for every $x, y \in R^\gamma$. If $x, y \in R^\gamma$ then $\varphi(xy) = xy(\gamma) = \sum_{\alpha\beta=\gamma} x(\alpha)y(\beta)$. Now $x(\alpha)y(\beta) = 0$ if either $\alpha \notin \text{Supp}(x)$ or $\beta \notin \text{Supp}(y)$. But $\alpha \in \text{Supp}(x)$ implies that $\alpha \leq \gamma$ and $\beta \in \text{Supp}(y)$ implies that $\beta \leq \gamma$. Hence $x(\alpha)y(\beta) \neq 0$ implies that $\gamma = \alpha\beta \leq \alpha \leq \gamma$. We conclude that $\alpha = \gamma$ and similarly $\beta = \gamma$. Therefore $\varphi(xy) = xy(\gamma) = x(\gamma)y(\gamma) = \varphi(x)\varphi(y)$ and hence φ is a homomorphism.

3. **π -semisimplicity.** Let π be a property of rings. We say R is a π -ring if it has property π . An ideal of a ring is said to be a π -ideal if as a ring it has property π .

We assume henceforth that the property π satisfies: (i) homomorphic images of π -rings are π -rings, and (ii) ideals of π -rings are π -rings. For example, the properties of being nil, left quasi-regular, locally nilpotent, nilpotent are such properties. A ring is said to be π -semisimple if it has no π -ideals.

LEMMA 2. *Let $R = \sum R_\alpha$ be a supplementary semilattice sum where R_α is π -semisimple for every $\alpha \in \Omega$. Suppose I is a π -ideal of R and let $x \in I$. Then $\text{Supp}(x)$ does not contain a maximum element with respect to the natural partial order on Ω .*

PROOF. Let $x = x(\alpha_1) + \dots + x(\alpha_n) \in I$ with $\alpha_n \geq \alpha_i, i = 1, \dots, n$. We will derive a contradiction. Since $\alpha_n \geq \alpha_i$ for $i = 1, \dots, n$, we have that $x \in R^{\alpha_n}$. Let $J = R^{\alpha_n} \cap I$. Then $x \in J$ and J is an ideal of I since R^{α_n} is an ideal of R . Hence J is a π -ring. Let $\varphi: R^{\alpha_n} \cap R_{\alpha_n}$ be the epimorphism defined in Lemma 1. Now J is an ideal of R^{α_n} , so $\varphi(J)$ is a π -ideal of R_{α_n} . Since $0 \neq x(\alpha_n) \in \varphi(J)$, $\varphi(J)$ is nonzero. This is the desired contradiction.

The main theorem on semisimplicity is proven for rings which are supplementary semilattice sums of subrings having the following property:

(A) For every $x \neq 0 \in R$, there exist $x', \check{x}'' \in R$ such that $0 \neq xx' = xx'x''$.

Any ring with an idempotent which is not a zero divisor satisfies (A).

THEOREM 1. *Suppose $R = \sum_{\alpha \in \Omega} R_\alpha$ is a supplementary semilattice sum where each R_α satisfies condition (A). If R_α is π -semisimple for every $\alpha \in \Omega$ then R is π -semisimple.*

PROOF. Suppose I is a π -ideal of R . Choose $x \in I, x \neq 0$, such that $\Gamma(x)$ is minimal. Say $x = x(\alpha_1) + \dots + x(\alpha_n)$ with $x(\alpha_i) \neq 0, i = 1, \dots, n$. Fix j such that $1 \leq j \leq n$ and let

$$y = x(\alpha_1)x(\alpha_1)' + \dots + x(\alpha_n)x(\alpha_n)' - x(\alpha_1)x(\alpha_1)'x(\alpha_j)'' - \dots - x(\alpha_n)x(\alpha_n)'(\alpha_j)''$$

where $x(\alpha_i)', x(\alpha_1)''$ are as hypothesized by condition (A). Now $\text{Supp}(y) \subseteq \langle \alpha_1, \dots, \alpha_n \rangle = \Gamma(x)$ so $\Gamma(y) \subseteq \Gamma(x)$. Also $x(\alpha_i)x(\alpha_i)' \neq 0$ for $i = 1, \dots, n$ and $x(\alpha_i)x(\alpha_i)'x(\alpha_j)'' \in R_{\alpha_i \alpha_j}$, so if $y = 0$ then for every $i = 1, \dots, n$ there exists i' such that $\alpha_i' \alpha_j = \alpha_i$. This implies $\alpha_i \alpha_j = \alpha_i$ which says that α_j is a maximum element in $\text{Supp}(x)$ contrary to Lemma 2. Therefore $y \neq 0$ and

by minimality of $\Gamma(x)$ we have $\Gamma(y)=\Gamma(x)$. Hence

$$\alpha_j \in \Gamma(x) = \Gamma(y) \\ = \langle \alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_n, \alpha_1\alpha_j, \dots, \alpha_{j-1}\alpha_j, \alpha_{j+1}\alpha_j, \dots, \alpha_n\alpha_j \rangle''$$

Therefore there exists $k_j \neq j$ such that $\alpha_j = \alpha_{k_j}\alpha_j$ or in other words $\alpha_j < \alpha_{k_j}$. Since the above argument works for every $j=1, \dots, n$, we can apply it to k_j and successively to each greater element that is obtained. Since $\text{Supp}(x)$ is finite we would eventually obtain elements α_k, α_l such that $\alpha_k < \alpha_l < \alpha_k$. This contradiction establishes the theorem.

In general the converse is not true. Let π be the property right-quasi regular. Then R is π -semisimple if and only if R is Jacobson semisimple. Let T be any integral domain which is not Jacobson semisimple (for example let T be the power series ring in one indeterminate over a field) and let Q be the quotient field of T . Let R be the subring of the ring of countably infinite row finite matrices over Q consisting of all matrices A for which there exists n such that, for $i, j > n$,

$$A_{ij} = 0, \quad i \neq j, \\ A_{ij} = a \in T, \quad \text{if } i = j.$$

Let P be the ideal of R consisting of all matrices A for which there exists n such that if $i, j > n$, $A_{ij}=0$. P is a prime ideal. Let S be the subring of R consisting of all matrices A for which $A_{ij}=0$, $i \neq j$ and $A_{ii}=a \in T$ for all positive integers i, j . Then R is a supplementary semilattice sum of the subrings S and P . But $S \cong T$ so S is not Jacobson semisimple. However R is a primitive ring (see [4, p. 36]) so certainly semisimple.

We observe without proof the following: Suppose that $R = \sum_{\alpha \in \Omega} R_\alpha$ is a supplementary semilattice sum of subrings R_α where each R_α has an identity element which lies in the center of R . If for every $\alpha \in \Omega$ there are no nonzero nilpotent elements in R_α , then there are no nonzero nilpotent elements in R . The proof is similar to that of Theorem 1.

4. Semigroup rings. If D is a semigroup and R is a ring, the *semigroup ring of D over R* , written RD , is defined to be the set of all functions from D into R which are zero off a finite set. We write elements of RD as finite formal sums $x = r_1d_1 + \dots + r_nd_n$ where $x(d_i) = r_i \in R$. Addition is point-wise and multiplication is convolution. Thus if $x, y \in RD$ then $(x+y)(d) = x(d)+y(d)$ and $xy(d) = \sum_{ab=d} x(a)y(b)$. Under these operations RD is a ring.

Note that if the semigroup D has a zero z we will not identify z with the zero of the semigroup ring RD . Thus $rz=0$ if and only if $r=0$ in R . Semigroup rings have been studied in [1], [5], [8] and [10].

A semigroup D is a *semilattice (band) of semigroups* $D_\alpha, \alpha \in \Omega$, if Ω is a semilattice (band), D is a disjoint union of the subsemigroups $D_\alpha, \alpha \in \Omega$, and $D_\alpha D_\beta \subseteq D_{\alpha\beta}$, for all $\alpha, \beta \in \Omega$. Semilattice decompositions of semigroups have been extensively studied (see [1], [6], [7], [9]). Observe that if D is a semilattice of semigroups $D_\alpha, \alpha \in \Omega$, then the semigroup ring RD is a supplementary semilattice sum of the semigroup ring $RD_\alpha, \alpha \in \Omega$. Thus we have the following corollary to Theorem 1.

COROLLARY 1. *Let R be a ring satisfying condition (A). Suppose D is a semilattice of semigroups $D_\alpha, \alpha \in \Omega$, where each D_α has an identity. If RD_α is π -semisimple, for every $\alpha \in \Omega$ then RD is π -semisimple.*

PROOF. If R satisfies (A) and D_α has an identity then RD_α satisfies (A).

Corollary 1 applies in particular to the case when R has an identity and D is a semilattice of groups. It would be interesting to know if the converse holds when D is a semilattice of groups. We can answer the question for a wide class of properties, but only in the very special case when the groups have one element; i.e., when D is a semilattice.

LEMMA 3. *Suppose π is a property such that R/I and I being π -rings imply that R is a π -ring. Let $R = \sum_{\alpha \in \Omega} R_\alpha$ be a supplementary semilattice sum, where Ω is a finite semilattice. If each R_α is a π -ring then R is a π -ring.*

PROOF. In the following paper [3, Lemma 3] we show that there exist disjoint nonempty subsemilattices A, B of Ω with $\Omega = A \cup B$ such that $R_A = \sum_{\alpha \in A} R_\alpha, R_B = \sum_{\beta \in B} R_\beta$ and $R = R_A + R_B$ are all supplementary semilattice sums where $A^2 = A, B^2 = B = AB = BA$. So an induction argument will complete the proof once we prove the theorem when $|\Omega| = 2$. So let $T = T_\alpha + T_\beta$ be a supplementary semilattice sum. Then since $\{\alpha, \beta\}$ is a semilattice either $\alpha\beta = \beta\alpha = \alpha$ or $\alpha\beta = \beta\alpha = \beta$. Suppose $\alpha\beta = \beta\alpha = \beta$. Then T_β is an ideal of T . Hence $T/T_\beta \cong T_\alpha/T_\alpha \cap T_\beta$. Therefore T/T_β being a homomorphic image of a π -ring is a π -ring. Since T_β is a π -ring, we conclude that T is a π -ring. Consider the following condition on properties of rings:

(F) A supplementary semilattice sum of π -rings $\sum_{\alpha \in \Omega} R_\alpha$ over Ω is a π -ring if, for every finite subsemilattice $\Omega' \subseteq \Omega$, the supplementary semilattice sum $\sum_{\alpha \in \Omega'} R_\alpha$ is a π -ring.

The properties of right quasi-regular, nil and regular satisfy (F).

THEOREM 2. *Assume π satisfies (F) and the hypothesis of Lemma 3. Let Ω be a semilattice. If the semigroup ring $R\Omega$ is π -semisimple then R is π -semisimple.*

PROOF. $R\Omega$ is a supplementary semilattice sum of the subrings $R\alpha = \{R\alpha \mid \Omega \in R\}$, $\alpha \in \Omega$. If R has a π -ideal P , then by Lemma 3 and condition (F), $P\Omega = \sum_{\alpha \in \Omega} P\alpha$ is a π -ideal of $R\Omega$. Hence $P = \{0\}$.

COROLLARY 2. *Assume π is as in Theorem 2. Let Ω be a semilattice and R a ring with identity. Then $R\Omega$ is π -semisimple if and only if R is π -semisimple.*

We now prove a generalization of Theorem 5.21 in [1]. If D is a semigroup, D^1 will denote the semigroup obtained from D by adjoining an identity element. Recall that if a semigroup D has a zero, rz considered as an element of the semigroup ring RD is zero if and only if $r=0$ in the ring R .

LEMMA 4. *Suppose D is an ideal extension of a group G by a commutative nil semigroup. If RD has no nilpotent ideals then D is a group.*

PROOF. We first show that D is separative ($a^2=ab=ba=b^2$ implies $a=b$). Let e be the identity of G . Then $ed=de$, for every $d \in D$. Assume that $a, b \in D$ with $a^2=ab=ba=b^2$. Then $(ae)^2=(ae)(be)=(be)^2$ and $ae, be \in G$. Hence $ae=be$. Therefore for every $g \in G$, $ag=bg$ and similarly $ga=gb$. Since D/G is commutative, then $cd \notin G$ implies $cd=dc$. Now consider $a-b \in RD$. We show that the principal ideal I generated by $a-b$ is nilpotent. In fact, $I^2=\{0\}$. It suffices to show that $(a-b)d(a-b)=0$ for all $d \in D^1$. If $d=1$, it is clear. Let $y=(a-b)d(a-b)=ada-bda-adb+bdb$. If ad, bd are both in G , then $(ad)a=(ad)b$ and $(bd)a=(bd)b$. So $y=0$. Suppose one of ad, bd is not in G . Without loss of generality assume $ad \in G$ and $bd \notin G$. Then $(ad)a=(ad)b$ and $(bd)a=dba=db^2=bdb$. So $y=0$.

A ring is *semiprime* if it has no nonzero nilpotent ideals. A commutative ring is semiprime if and only if it has no nil ideals.

THEOREM 3. *Suppose R is a commutative ring with identity and that D is a commutative semigroup such that a power of each element lies in a subgroup. Then RD is semiprime if and only if D is a semilattice of groups G_α , $\alpha \in \Omega$, and RG_α is semiprime for every $\alpha \in \Omega$.*

PROOF. Suppose RD is semiprime. If D is as stated then D is a semilattice of semigroups D_α , $\alpha \in \Omega$, where each D_α contains a group-ideal G_α with D_α/G_α a nil semigroup (see [1, §4.3, Exercise 5]). Now if RD_α had a nilpotent ideal N_α , then since RD is commutative, N_α would generate a nilpotent ideal of RD . Hence, for every $\alpha \in \Omega$, RD_α has no nonzero nilpotent ideal and, by Lemma 4, D_α is a group. The converse follows from Theorem 1.

REMARK. Let D be a union of groups and F a field. In [4], Munn proved that if D is finite then FD semisimple implies D is a semilattice of

groups. The preceding theorem shows that the same conclusion holds if D is commutative and FD is semiprime. Janeski and Weissglass have proven the identical conclusion under the assumption that FD is regular (without assuming either commutativity or finiteness). One might be led to conjecture that if D is a union of groups and FD is Jacobson semisimple then D is a semilattice of groups. This is false. Let $D = \mathcal{M}(G^0, 2, 2, P)$ be the Rees 2×2 matrix semigroup over the group with zero G^0 with sandwich matrix P (cf. [1, p. 88]) where G is the infinite cyclic group with generator a and

$$P = \begin{bmatrix} 1 & a \\ 1 & 1 \end{bmatrix}.$$

By applying Theorem 3.8 of [10], QD is Jacobson semisimple. But D is a union of groups (in fact D is a completely simple semigroup) which is not a semilattice of groups.

Questions. 1. Find a condition on Ω such that if R is the supplementary semilattice sum of R_α , $\alpha \in \Omega$, then R π -semisimple implies R_α is π -semisimple.

2. If $R = \sum R_\alpha$, $\alpha \in \Omega$, is a supplementary semilattice sum and R is commutative, does R Jacobson semisimple imply that each R_α is Jacobson semisimple?

3. Can any of the results in this article be generalized to the case where Ω is a band?

4. Can condition (A) be removed in Theorem 1?

5. Find necessary and sufficient conditions that a ring be a supplementary semilattice sum of division rings.

6. Find conditions to insure that a ring be a supplementary semilattice sum of semilattice sum indecomposable rings. Characterize semilattice sum indecomposable rings.

7. Let $D = \bigcup_{\alpha \in \Omega} G_\alpha$ be a semilattice of groups. Does the Jacobson semisimplicity of RD imply the Jacobson semisimplicity of RG_α for every $\alpha \in \Omega$?

8. Find a property π of rings such that if D is any semigroup then RD is a π -ring if and only if D is a semilattice of groups G_α and RG_α is a π -ring, for every $\alpha \in \Omega$.

9. Can the commutativity of R be removed in Theorem 3?

The author would like to thank Mohan Putcha for many helpful discussions.

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