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COHOMOLOGY OF A BOUNDING MANIFOLD

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ABSTRACT. This paper characterizes those subsets of $H^*(M; Q)$ which are the image of the cohomology of some manifold whose boundary is M.

1. **Introduction.** The object of this note is to prove:

PROPOSITION. *Let M^n be a closed oriented (or stably almost complex) manifold and $R^* \subset H^*(M; Q)$. There is a compact oriented (or s.a.cx.) manifold with boundary V so that $\partial V = M$ with $R^* = i^*H^*(V; Q)$ where $i: M \to V$ is the inclusion if and only if R^* satisfies:*

(a) *R^* is a homogeneous subalgebra of $H^*(M; Q)$ (i.e. R^* is the direct sum of the groups $R^i = R^* \cap H^i(M; Q)$);*

(b) *R^* contains the characteristic ring $\tau_M^* H^*(BSO; Q)$ (or $\tau_M^* H^*(BU; Q)$) where $\tau_M: M \to BSO$ (or BU) classifies the tangent bundle of M;*

(c) *R^* is its own annihilator $R^{*\perp}$, where*

$$R^{*\perp} = \{x \in H^*(M; Q) \mid \langle x \cup y, [M] \rangle = 0 \quad \forall y \in R^*\};$$

and

(d) *(in the oriented case only) if $M' \subset M$ is an open-closed submanifold (union of components) with the unit class $1_{M'} \in R^0$, then M' is an unoriented boundary.*

The analogous result for mod 2 cohomology of closed manifolds was proved in [**1**].

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2. **Proof of the Proposition.** It is easy to see that $i^*H^*(V; Q)$ satisfies (a)–(d). Specifically, (b) follows from $\tau_M = \tau_V \circ i$, (c) is a standard consequence of Poincaré-Lefschetz duality, and (d) is obtained by noting that each M' is the boundary of some union of components of V. The other implication is the difficult one.

LEMMA. *Let X be a finite complex and $R^* \subset H^*(X; Q)$ a homogeneous subalgebra with unit. Then X decomposes uniquely into the disjoint union of*

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open-closed subsets $X_1 \cup \cdots \cup X_r$ *so that* R° *has a basis given by the unit classes* $1_{X_1}, \cdots, 1_{X_r}$.

PROOF. Suppose X is the disjoint union $Y_1 \cup \cdots \cup Y_s$ where each Y_i is an open-closed subset of X and $e = \alpha_1 1_{Y_1} + \cdots + \alpha_s 1_{Y_s} \in R^\circ$ where the α_i are distinct rationals. Then $e^p = \alpha_1^p 1_{Y_1} + \cdots + \alpha_s^p 1_{Y_s}$ is in R° for $p = 0, 1, \cdots, s-1$. Since the Vandermonde determinant is nonsingular, each 1_{Y_i} lies in the span of $e^\circ = 1, e, \cdots, e^{s-1}$, and hence in R°. □

LEMMA. *Let X be a finite complex and $R^* \subset H^*(X; Q)$ a homogeneous subalgebra with unit. There is a torsion free space Y and a continuous map $f: X \to Y$ so that $f^*H^*(Y; Q) = R^*$ and so that $f^*: H^i(Y; Z_2) \to H^i(X; Z_2)$ is the zero homomorphism for all $i > 0$.*

PROOF. By the previous lemma, one may decompose X into $X_1 \cup \cdots \cup X_r$, with $\{1_{X_i}\}$ a base for R°. Then $1_{X_i} \cdot R^* = R^* \cap H^*(X_i; Q)$ decomposes R^* as a direct sum. If one finds appropriate maps for each X_i to realize $1_{X_i} \cdot R^*$, the disjoint union will give the desired map for X. Thus, one may suppose $R^\circ \simeq Q$ with base the unit class 1. Let n be the dimension of X. For any class $x \in H^j(X; Q)$, $j > 0$, there is an integer $m_x \neq 0$ and class $x' \in H^j(X; Z)$ so that $\rho_Q x' = m_x x$, where ρ_Q denotes rational reduction. Letting $s > n$, one may find a map $g_x: X \to K(Z, j)^s$ into the s-skeleton of an Eilenberg-Mac Lane space so that $g_x^*(i_j) = 2x'$, where i_j denotes the fundamental class.

There is a class $\sigma_j \in K^*(K(Z, j)^s)$ for which $\text{ch}(\sigma_j) = p_x \rho_Q(i_j) +$ higher terms, where $p_x \neq 0$ is an integer, ch being the Chern character. In particular, there is a map $\phi_j: K(Z, j)^s \to Y_j$, where $Y_j = BU$ for j even or U for j odd so that $\phi_j^*(\text{ch}_j) = p_x \rho_Q(i_j)$, where $\text{ch}_j \in H^j(Y_j; Q)$ is the j-dimensional component of the Chern character.

Now let $x_i \in H^{j_i}(X; Q)$, $j_i > 0$, be a set of generators for R^*. Let Y be the product $\prod Y_{j_i}$ of the corresponding Y_j's and $f: X \to Y$ the product $\prod \phi_{j_i} \circ g_{x_i}$ of the maps corresponding to x_i. Y is connected, since each Y_{j_i} is connected so $f^*H^\circ(Y; Q) = R^\circ$ and similarly Y is torsion free. Since $g_{x_i}^*(i_{j_i}) = 2x'$, $g_{x_i}^*$ is zero on positive dimensional mod 2 cohomology, and hence f^* is also. Then $(\phi_{j_i} \circ g_{x_i})^*(\text{ch}_{j_i}) = 2p_{x_i} m_{x_i} \cdot x_i$, so $f^*H^*(Y; Q) \supset R^*$ and $\text{im} f^* \subset \text{im}(\prod g_{x_i})^* \subset R^*$ so $f^*H^*(Y; Z) = R$. □

Now let M^n be a closed oriented (or s.a.cx.) manifold, with $R^* \subset H^*(M; Q)$ satisfying conditions (a)–(d). Let $f: M \to Y$ be the map given by the Lemma so that $f^*H^*(Y; Q) = R^*$.

For $u \in H^j(Y; Q)$ and $v \in H^{n-j}(BSO; Q)$ (or $H^{n-j}(BU; Q)$), $f^*(u) \in R^*$ and $\tau_M^*(v) \in R^* = R^{*\perp}$, and so $\langle \tau_M^*(v) \cup f^*(u), [M] \rangle = 0$, so that all Pontrjagin (or Chern) numbers of $f: M \to Y$ are zero. Further, in the oriented case, for all $u \in H^j(Y; Z_2)$ and $v \in H^{n-j}(BSO; Z_2)$, $\langle \tau_M^*(v) \cup f^*(u), [M] \rangle = 0$

(if $j>0$, $f^*(u)=0$, while for $j=0$, $f^*(u)$ is a sum of classes 1_{M_i}, where $M=M_1\cup\cdots\cup M_r$ is the decomposition given by R° and this is a Stiefel-Whitney number of the corresponding union of the M_i—which is zero since each M_i is an unoriented boundary), and so all Stiefel-Whitney numbers of $f: M\to Y$ are zero.

Since Y is torsion free, characteristic numbers determine the bordism class, and $[M, f]$ is zero in $\Omega_n^{SO}(Y)$ (or $\Omega_n^U(Y)$). Thus, there is a compact oriented (or s.a.cx.) manifold with boundary V having $\partial V=M$ and map $F: V\to Y$ so that $f=F\circ i$.

Then $R^*=f^*H^*(Y; Q)=i^*F^*H^*(Y; Q)\subset i^*H^*(V; Q)$ and taking annihilators, $R^*=R^{*\perp}\supset i^*H^*(V; Q)^\perp=i^*H^*(V; Q)$. Thus V is the required manifold with boundary. □

Reference

1. R. E. Stong, *Bilinear forms on manifolds*, Math. Ann. (to appear).

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