PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 3, August 1973

A CHARACTERIZATION OF THIN OPERATORS IN A VON NEUMANN ALGEBRA[1]

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ABSTRACT. Let $\mathscr{A}$ be a von Neumann algebra, $\mathscr{I}$ a uniformly closed, weakly dense, two-sided ideal in $\mathscr{A}$, $\mathscr{Z}$ the center of $\mathscr{A}$, and $\mathscr{P}$ the lattice of projections in $\mathscr{I}$. An operator $A \in \mathscr{A}$ is *thin* relative to $\mathscr{I}$ if $A=Z+K$, for some $Z \in \mathscr{Z}$, $K \in \mathscr{I}$. The thin operators relative to $\mathscr{I}$ are characterized as those $A \in \mathscr{A}$ satisfying $\lim_{P \in \mathscr{P}} \|AP-PA\|=0$. It is also shown that

$$\limsup_{P \in \mathscr{P}} \|PAP - AP\| = \limsup_{P \in \mathscr{P}} \|PAP - PA\|.$$

1. Let $\mathscr{A}$ be a von Neumann algebra, $\mathscr{I}$ a uniformly closed, weakly dense, two-sided ideal in $\mathscr{A}$, $\mathscr{Z}$ the center of $\mathscr{A}$, and $\mathscr{P}$ the lattice of projections in $\mathscr{I}$. An operator $A \in \mathscr{A}$ is *thin* relative to $\mathscr{I}$ if $A=Z+K$, for some $Z \in \mathscr{Z}$ and $K \in \mathscr{I}$. The set $\mathscr{P}$ is directed under the usual ordering, and for a fixed $A \in \mathscr{A}$, the map $P \to \|AP-PA\|$ is a net on $\mathscr{P}$. In Theorem 1 we characterize the thin operators relative to $\mathscr{I}$ as those $A \in \mathscr{A}$ satisfying

$$\lim_{P \in \mathscr{P}} \|AP - PA\| = 0.$$

This discussion is a sequel to a previous paper [7] in which a version of this characterization was demonstrated for the case of $\mathscr{A}$ a factor. The proof to be presented here requires results of the previous paper.

It was proved in [7] that for $\mathscr{A}$ a factor, $A \in \mathscr{A}$ is thin relative to $\mathscr{I}$ if and only if

$$\lim_{P \in \mathscr{P}} \|PAP - AP\| = 0.$$

This was done by R. G. Douglas and Carl Pearcy for $\mathscr{A}=\mathscr{B}(\mathscr{H})$, the algebra of all bounded operators on a separable Hilbert space $\mathscr{H}$, and $\mathscr{I}$ the ideal of compact operators [3]. It is an immediate corollary that for

Presented to the Society, April 13, 1972; received by the editors May 30, 1972 and, in revised form, November 21, 1972.

AMS (MOS) subject classifications (1970). Primary 47C15; Secondary 48B39, 47B05, 46L10.

[1] Research supported by NRC Grants A7169 and A5211.

$\mathscr{A}$ a factor, $A \in \mathscr{A}$ is thin if and only if

$$\lim_{P \in \mathscr{P}} \|AP - PA\| = 0.$$

This symmetric formulation is a more natural one to use in describing the thin operators. It is obviously selfadjoint and is perhaps suggestive of the fact that $A \in \mathscr{A}$ is thin if and only if the range of the inner derivation on $\mathscr{A}$ induced by A is contained in $\mathscr{I}$ [**2**, p. 259].

The original formulation was suggested by P. R. Halmos for $\mathscr{A} = \mathscr{B}(\mathscr{H})$, and $\mathscr{I}$ = compact operators; it is a strengthening of the defining condition for quasitriangularity. An operator $A \in \mathscr{B}(\mathscr{H})$ is quasitriangular if

$$\liminf_{P \in \mathscr{P}} \|PAP - AP\| = 0,$$

where $\mathscr{P}$ is the finite rank projections. An operator A may be quasitriangular when A^* is not; thus it is not in general true that

$$\liminf_{P \in \mathscr{P}} \|PAP - AP\| = \liminf_{P \in \mathscr{P}} \|PAP - PA\|$$

(see [**4**]). However, we show in Theorem 2 that

$$\limsup_{P \in \mathscr{P}} \|PAP - AP\| = \limsup_{P \in \mathscr{P}} \|PAP - PA\| = \limsup_{P \in \mathscr{P}} \|AP - PA\|.$$

When $\mathscr{A} = \mathscr{B}(\mathscr{H})$ and $\mathscr{I}$ = compact operators, this follows from a result of Norberto Salinas [**8**, Theorem 3]. We conjecture that

$$\limsup_{P \in \mathscr{P}} \|AP - PA\| = \inf_{Z \in \mathscr{Z}, K \in \mathscr{I}} \|A + Z + K\|.$$

That is, if $\pi: \mathscr{A} \to \mathscr{A}/\mathscr{I}$ is the natural quotient map, then

$$\limsup_{P \in \mathscr{P}} \|AP - PA\|$$

is the distance of $\pi(A)$ from $\pi(\mathscr{Z})$, the center of the C^*-algebra $\mathscr{A}/\mathscr{I}$ [**2**, p. 259].

The author wishes to thank James A. Deddens for helpful suggestions on this work.

2. The following proposition was essentially established in [**7**].

PROPOSITION. *Let $\mathscr{A}$, $\mathscr{I}$, $\mathscr{Z}$, $\mathscr{P}$ be as above. If $A \in \mathscr{A}$ is thin relative to $\mathscr{I}$, then*

$$\lim_{P \in \mathscr{P}} \|AP - PA\| = 0.$$

PROOF. It follows by Proposition 2.1 of [7] that

$$\lim_{P\in\mathscr{P}} \|PAP - AP\| = 0.$$

Since A thin implies A^* thin, it follows that

$$\lim_{P\in\mathscr{P}} \|PA^*P - A^*P\| = \lim_{P\in\mathscr{P}} \|PAP - PA\| = 0,$$

and hence that

$$\lim_{P\in\mathscr{P}} \|AP - PA\| = 0.$$

To show the converse, we require two lemmas.

LEMMA 1. *Let $\mathscr{A}$, $\mathscr{I}$, $\mathscr{Z}$, $\mathscr{P}$ be as above. For $A \in \mathscr{A}$, assume there is an irreducible representation φ of $\mathscr{A}$ such that $\varphi(\mathscr{I})\neq 0$ and $\varphi(A+K+\lambda)\neq 0$, for any $K \in \mathscr{I}$, any complex scalar λ. Then*

$$\limsup \|AP - PA\| > 0.$$

PROOF. Using Proposition 2.2 of [7] we see that

$$\inf_{P\in\mathscr{P}} \eta_{\varphi(A)}(\varphi(I-P)\mathscr{H}) \leqq \limsup_{P\in\mathscr{P}} \|PA(I-P)\| \leqq \limsup_{P\in\mathscr{P}} \|AP - PA\|,$$

where

$$\eta_{\varphi(A)}(\varphi(I-P)\mathscr{H}) = \sup_{x\in\varphi(I-P)\mathscr{H}:\|x\|=1} \|\varphi(A)x - (\varphi(A)x, x)x\|.$$

Now if

$$\inf_{P\in\mathscr{P}} \eta_{\varphi(A)}(\varphi(I-P)\mathscr{H}) = 0,$$

then, as in the proof of Theorem 2.3 of [7], we obtain an operator $K \in \mathscr{I}$ and a scalar λ with $\varphi(A+K+\lambda)=0$. Thus the conclusion follows.

LEMMA 2. *Let $\mathscr{A}$, $\mathscr{I}$, $\mathscr{Z}$, $\mathscr{P}$ be as above. If $A \in \mathscr{A}$ is selfadjoint and has the property that $\|AP-PA\|<\varepsilon$ for all $P \in \mathscr{P}$, then there is a $Z \in \mathscr{Z}$ with $\|A-Z\|<3\varepsilon$.*

PROOF. Observe that the strongly closed convex hull of $\mathscr{P}$ is the set of all positive operators in the unit ball of $\mathscr{A}$. For, this set of positive operators is convex and weakly compact, hence it is the weakly closed convex hull of its extreme points, which are the projections of $\mathscr{A}$. Furthermore, $\mathscr{P}$ is weakly dense in the projections of $\mathscr{A}$.

Assume that $A \in \mathscr{A}$ is selfadjoint and satisfies $\|AP-PA\|\leqq\varepsilon$ for all $P \in \mathscr{P}$. If T is a convex combination of projections in $\mathscr{P}$, then also $\|AT-TA\|\leqq\varepsilon$, and similarly for T a strong limit of such convex combinations. Thus if T is any positive operator in $\mathscr{A}$ of norm at most 1, then

$\|AT-TA\| \leqq \varepsilon$. Now, for arbitrary $T \in \mathscr{A}$ with $\|T\| \leqq 1$, we can write $T=B_1-B_2+i(B_3-B_4)$, a linear combination of positive operators $B_i \in \mathscr{A}$ with $\|B_i\| \leqq 1$, and thereby deduce that $\|AT-TA\| \leqq 4\varepsilon$.

Thus we have shown that the inner derivation $\mathscr{D}_A$ on $\mathscr{A}$, given by $\mathscr{D}_A: T \to AT-TA$ has norm $\|\mathscr{D}_A\| \leqq 4\varepsilon$. Since A is selfadjoint, it follows from a result of R. V. Kadison [6] that

$$\|\mathscr{D}_A\| = 2 \inf_{Z \in \mathscr{Z}} \|A - Z\|.$$

Thus we can find $Z \in \mathscr{Z}$ with $\|A-Z\| < 3\varepsilon$. The lemma is proved.

THEOREM 1. *Let $\mathscr{A}$ be a von Neumann algebra, $\mathscr{I}$ a uniformly closed, weakly dense ideal in $\mathscr{A}$ and $\mathscr{P}$ the lattice of projections in $\mathscr{I}$. Then $A \in \mathscr{A}$ is thin relative to $\mathscr{I}$ if and only if*

$$\lim_{P \in \mathscr{P}} \|AP - PA\| = 0.$$

PROOF. In view of the proposition, it suffices to assume $A \in \mathscr{A}$ is not thin and show that

$$\limsup_{P \in \mathscr{P}} \|AP - PA\| > 0.$$

It is easy to see that it is sufficient to prove this theorem for A selfadjoint, and we will later restrict to such A.

Let $A \in \mathscr{A}$ be such that $A \neq Z+K$, any $Z \in \mathscr{Z}$, $K \in \mathscr{I}$. Let Ω be the maximal ideal space of $\mathscr{Z}$ and identify $\mathscr{Z}$ with $C(\Omega)$. The proof proceeds as follows: a maximal ideal $\gamma \in \Omega$ is located such that $A \neq Z+K$ relative to the ideal $[\gamma]$ in $\mathscr{A}$, where $[\gamma]$ is generated by γ. Then the cases $\mathscr{I} \not\subset [\gamma]$ and $\mathscr{I} \subset [\gamma]$ are considered separately.

Since $\mathscr{Z}+\mathscr{I}$ is closed, $\exists \varepsilon > 0$ with $\|A+Z+K\| \geqq \varepsilon$, all $Z+K$. Set $E_0=I$. If $\mathscr{A}$ is not a factor, there is some nontrivial projection $E_1 \in \mathscr{Z}$ with $\|(A+Z+K)E_1\| \geqq \varepsilon$ for all $Z \in \mathscr{Z}$, $K \in \mathscr{I}$. For, if not, let P_1 be any nontrivial central projection and set $P_2=I-P_1$. Then there exists $Z_i \in \mathscr{Z}$, $K_i \in \mathscr{I}$ with $\|(A+Z_i+K_i)P_i\| < \varepsilon$, $i=1, 2$. But then, for $Z=Z_1P_1+Z_2P_2$ and $K=K_1P_1+K_2P_2$, we have $\|A+Z+K\| < \varepsilon$, a contradiction. Continuing, if $E_1\mathscr{A}$ is not a factor, we can find a suitable $E_2 < E_1$, and so on. In this way, a strictly decreasing sequence of central projections $\{E_n\}$ is chosen with $\|(A+Z+K)E_n\| \geqq \varepsilon$ for all $Z \in \mathscr{Z}$, $K \in \mathscr{I}$, for each n.

Observe that E_n corresponds to a characteristic function $\chi_{\Omega_n} \in C(\Omega)$, where $\{\Omega_n\}$ is a nested sequence of open and closed sets. Set $\Gamma = \bigcap \Omega_n$, a compact set. (Note that if the sequence $\{E_n\}$ is finite, Γ is an isolated point.) For each $\omega \in \Omega$, let $[\omega]$ be the two-sided uniformly closed ideal in $\mathscr{A}$ generated by ω. It is a result of H. Halpern that each ideal $[\omega]$ is

primitive [5, p. 213, Theorem 4.7]. Let $\{U_\alpha\}$ be a fundamental system of open and closed neighborhoods of ω in Ω, and let F_α be the projection corresponding to $\chi_{U\alpha}$. Then we can characterize

$$[\omega] = \{T \in \mathscr{A} : \|TF_\alpha\| \underset{\alpha}{\longrightarrow} 0\}.$$

In fact, observe that the set

$$\mathscr{S} = \{Q \in \mathscr{A} : Q \text{ a projection and } Q(I - F_\alpha) = Q \text{ for some } \alpha\}$$

is the smallest p-ideal containing the projections of ω. Hence $\mathscr{S}$ is precisely the set of all projections in $[\omega]$ (see [9]). On the other hand, the set on the right above is obviously a uniformly closed two-sided ideal containing exactly these same projections, so the equality follows [9].

We claim there is some point $\gamma \in \Gamma$ with $A+K+\lambda \notin [\gamma]$, for any $K \in \mathscr{I}$, any scalar λ. (If $\mathscr{A}$ is a factor we have $[\gamma]=0$.) Assume the contrary; that is, for each $\omega \in \Gamma$ there is a $K_\omega \in \mathscr{I}$ and a scalar λ_ω with $A+K_\omega+\lambda_\omega \in [\omega]$. Then there is an open and closed neighborhood U_ω of ω such that

$$\|(A + K_\omega + \lambda_\omega)F_\omega\| < \varepsilon/2.$$

The set $\{U_\omega\}$ is an open cover of Γ; reduce to a finite subcover $\{U_j\}$ with corresponding $\{F_j\}$, $\{K_j\}$, and $\{\lambda_j\}$, $j=1, 2, \cdots, m$. We may assume the U_j's are disjoint. Set $U=\bigcup U_j$, $F=\sum F_j$, $K=\sum K_jF_j$ and $Z=\sum \lambda_jF_j$. Then $\|(A+K+Z)F\|<\varepsilon/2$. Now, U is an open and closed neighborhood of $\Gamma=\bigcap \Omega_n$, a nested intersection of open and closed sets. An elementary topological argument shows that there is some $\Omega_k \subset U$. Then $E_k \leqq F$ and

$$\|(A + K + Z)E_k\| \leqq \|(A + K + Z)F\| < \varepsilon/2;$$

a contradiction to the original choice of E_k. Hence the claim is established.

Fix $\gamma \in \Gamma$ with $A+K+\lambda \notin [\gamma]$, any $K \in \mathscr{I}$, λ scalar. We consider two cases: $\mathscr{I} \not\subset [\gamma]$ and $\mathscr{I} \subset [\gamma]$. Suppose that $\mathscr{I} \not\subset [\gamma]$. Let φ be the irreducible representation of $\mathscr{A}$ with kernel $[\gamma]$. The hypotheses of Lemma 1 are satisfied, and we conclude that

$$\limsup_{P \in \mathscr{P}} \|AP - PA\| > 0,$$

which finishes this case.

Assume now that $\mathscr{I} \subset [\gamma]$. Note that $A+\lambda \notin [\gamma]$ for any scalar λ. We claim there is some $\delta>0$ such that: for each F_α corresponding to some fundamental neighborhood U_α of γ, there is a $P_\alpha \in \mathscr{P}$ with

$$\|(AP_\alpha - P_\alpha A)F_\alpha\| > \delta.$$

If this were not so, we could find a sequence α_n with $\|(AP-PA)F_{\alpha_n}\|<1/n$, all $P \in \mathscr{P}$. Consider then $\mathscr{I}F_{\alpha_n}$, a uniformly closed, weakly dense ideal in

$\mathscr{A}F_{\alpha_n}$. Since we may assume A to be selfadjoint, we can apply Lemma 2 to obtain Z_{α_n} in the center of $\mathscr{Z}F_{\alpha_n}$ with $\|(A-Z_{\alpha_n})F_{\alpha_n}\|<3/n$, for $n=1,2,\cdots$. Set $L_{\alpha_n}=(A-Z_{\alpha_n})(I-F_{\alpha_n})$, $n=1,2,\cdots$, so $L_{\alpha_n}\in[\gamma]$. Then considering Z_{α_n} to be in the center of $\mathscr{A}$, we get

$$\|A-(L_{\alpha_n}+Z_{\alpha_n})\| = \|(A-Z_{\alpha_n})F_{\alpha_n}\| < 3/n.$$

Thus $\{L_{\alpha_n}+Z_{\alpha_n}\}$ converges uniformly to A, so $A\in[\gamma]+\mathscr{Z}$. In other words, there is a $Z\in\mathscr{Z}$ with $A+Z\in[\gamma]$. Viewing Z as a function on Ω, we have $A+Z(\gamma)\in[\gamma]$ and $Z(\gamma)$ is a scalar; a contradiction. Thus the claim is established.

Fix $\delta>0$ such that, given α, there is a $P\in\mathscr{P}$ with $\|(AP-PA)F_\alpha\|>\delta$. To finish the proof, let an arbitrary $P_0\in\mathscr{P}$ be given. We find $Q>P_0$, $Q\in\mathscr{P}$, with $\|AQ-QA\|>\delta$. Since we are assuming $\mathscr{I}\subset[\gamma]$, there is some β with $P_0(I-F_\beta)=P_0$. There is some $P\in\mathscr{P}$ with $\|(AP-PA)F_\beta\|>\delta$; assume without loss of generality that $\|(PA(I-P))F_\beta\|>\delta$. Set $Q=P_0+PF_\beta$. Then $Q>P_0$, $Q\in\mathscr{P}$, and

$$\begin{aligned}\|AQ-QA\| &\geqq \|QA(I-Q)\| \geqq \|QA(I-Q)F_\beta\| \\ &= \|PA(I-P)F_\beta\| > \delta.\end{aligned}$$

Thus in the case $\mathscr{I}\subset[\gamma]$ we also have

$$\limsup_{P\in\mathscr{P}} \|AP-PA\| > \delta$$

and the proof is complete.

3. The proof of the following theorem is basically a generalization of the proof of Theorem 3 of [**8**].

THEOREM 2. *Let $\mathscr{A}$ be a von Neumann algebra, $\mathscr{I}$ a uniformly closed two-sided ideal of $\mathscr{A}$, and $\mathscr{P}$ the lattice of projections in $\mathscr{I}$. For $A\in\mathscr{A}$,*

$$\limsup_{P\in\mathscr{P}} \|(I-P)AP\| = \limsup_{P\in\mathscr{P}} \|PA(I-P)\| = \limsup_{P\in\mathscr{P}} \|AP-PA\|.$$

PROOF. The second equality follows from the first. In order to prove the first equality it suffices to show

$$\limsup_{P\in\mathscr{P}} \|PA(I-P)\| \geqq \limsup_{P\in\mathscr{P}} \|(I-P)AP\|$$

and then apply this to A^*.

Let $P_0\in\mathscr{P}$ and $\varepsilon>0$ be given. Then it suffices to find a $Q\in\mathscr{P}$ such that $Q\geqq P_0$ and

$$\|QA(I-Q)\| \geqq \limsup_{P\in\mathscr{P}} \|(I-P)AP\| - 3\varepsilon.$$

Now there is a $P_1 \in \mathscr{P}$ with $P_1 \geqq P_0$ and

$$\|(I - P_0)AP_0 - P_1(I - P_0)AP_0\| \leqq \varepsilon; \quad \text{i.e., } \|(I - P_1)AP_0\| \leqq \varepsilon.$$

For, we set $P_1 = P_0 \vee E((\varepsilon, \|A\|])$, where $E(\lambda)$ is the spectral resolution for the operator

$$T = [(I - P_0)AP_0((I - P_0)AP_0)^*]^{1/2},$$

[**1**, p. 855, Lemma 4.1]. Then we choose $P_2 \geqq P_1$ with $P_2 \in \mathscr{P}$ and

$$\|(I - P_2)AP_2\| > \limsup_{P \in \mathscr{P}} \|(I - P)AP\| - \varepsilon.$$

Finally, as with P_1, we can find $R \in \mathscr{P}$ with $R \leqq I - P_2$ and

$$\|(I - P_2)AP_2 - R(I - P_2)AP_2\| \leqq \varepsilon.$$

Set $Q = P_0 \vee R$; then $Q \in \mathscr{P}$. Note that $R = R(I - P_2) \leqq Q$, $P_2 - P_0 \leqq I - Q$, and $R(I - P_2) \leqq I - P_1$. Thus,

$$\begin{aligned}
\|QA(I - Q)\| &\geqq \|R(I - P_2)QA(I - Q)(P_2 - P_0)\| \\
&= \|R(I - P_2)A(P_2 - P_0)\| \\
&\geqq \|R(I - P_2)AP_2\| - \|R(I - P_2)AP_0\| \\
&\geqq \|R(I - P_2)AP_2\| - \|(I - P_1)AP_0\| \\
&\geqq \|(I - P_2)AP_2\| - 2\varepsilon \\
&\geqq \limsup_{P \in \mathscr{P}} \|(I - P)AP\| - 3\varepsilon.
\end{aligned}$$

The proof is finished.

ADDED IN PROOF. Similar results have been obtained by C. Apostol and L. Zsidó. They show that the conjecture holds in factors and for certain ideals in general von Neumann algebras.

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