HALVING THE MILNOR MANIFOLDS
AND SOME CONJECTURES OF RAY

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Abstract. The peculiar properties of the 2'^-2 dimensional
generators of unitary bordism (the 2-primary Milnor generators)
are related to the 2'^-3 dimensional indecomposable torsion classes
of Alexander. This result is then used to confirm a conjecture of
Ray concerning the generalised homology spectral sequence for
MSp*(MU). Finally it is noted that Ray's conjecture to the effect
that all classes in MSp* are detectable by KO-characteristic numbers
must fail.

1. Halving the Milnor manifolds. We consider the cofibration sequence
of spectra

\[ MSp \xrightarrow{i} MU \xrightarrow{p} MU/MSp; \]

\[ MU/MSp \] will be the Thom spectra for \((U, Sp)\) relative bordism. Recall
that \(MU_\# = \mathbb{Z}[x_1, x_2, \cdots, x_i, \cdots], x_i \in MU_{2i}\) and that for \(i=2^k-1\) the
\(x_i\) (the 2-primary Milnor generators) have the peculiarity that, if \(h_*\) is the
Hurewicz map, then \(h_*(x_i)\) is divisible by 2 in \(H_{2i}(MU; \mathbb{Z})\) although of
course \(x_i\) is not divisible in \(MU_\#\). We will now shed some light on this
peculiar behavior.

J. C. Alexander [1] has defined a series of indecomposable torsion ele-
ments \(\mu_i \in MSp_{8i-3}, 2\mu_i = 0\). Let \(\sigma_1\) be the generator of \(MSp\). In all that
follows \(n\) will be equal to \(2^k-2\). Let \(\delta: MU/MSp \rightarrow SMSGp\) be the cofibre of
\(p\). Since \(j_*\mu_n = 0\), we can choose \(\tilde{\mu}_n \in (MU/MSp)_{8n-2}\) such that \(\delta_*\tilde{\mu}_n = \mu_n\).
Let

\[ f_k: S^{8n-3} \rightarrow MSp; \quad g_k: S^{8n-2} \rightarrow MU/MSp \]

classify \(\mu_n\), \(\tilde{\mu}_n\) respectively; \(\partial \circ g_k \cong Sf_k\). Let \(U_2 \in H^0(MSp; \mathbb{Z})\) be the
Thom class and \(s' \in H^i(S'; \mathbb{Z})\) the generator. Then Alexander has shown
that \(f_k\) is detected by a functional cohomology operation, namely,

\[ Sq^{2a(k)}(U_2) = S^{8n-3} \]
or, suspending once,

$$Sq_{S/2}^{2\Delta(k)}(SU_2) = s^{8n-2}$$

But \( \partial^*(SU_2) = 0 \) so

$$g_k^* SQ_{S/2}^{2\Delta(k)}(SU_2) = s^{8n-2}$$

and \( g_k \) can be detected by cohomology in the ordinary way. \((Sq^{2\Delta(k)}\) denotes \( Sq_{(0,0,\ldots,0,2,\ldots)} \) \( 2 \) in the \( k \)th position) in the Milnor basis notation.) Recall the definition of functional operations and compute \( SQ_{S/2}^{2\Delta(k)}(SU_2) \):

$$S[Sq_{S/2}^{2\Delta(k)}(SU_2)] = (Sp)^* Sq_{S/2}^{2\Delta(k)}(Sp)^{-1} U_2.$$  

Now \( (Sp)^{-1}(U_2) \) is well defined; it is just \( \) (the suspension of) the mod 2 Thom class of \( MU \); call it \( U_2 \). \( Sq^{2\Delta(k)}U_2 \) is the primitive class in the coalgebra \( H^{8n-2}(MU; \mathbb{Z}) \). Thus taking normal Chern numbers on \( (MU/MSp)_* \) we have that \( s_{(4n-1)}(c)[\bar{\mu}_n] \) is odd. Since \( \mu_n = 0 \) we can pull \( 2\bar{\mu}_n \) back to \( MU_{8n-2} \) and the pull back can be chosen as \( x_{4n-1} \), a 2-primary Milnor generator. In other words, the 2-primary Milnor generators (properly chosen) can be halved by being regarded as classes in the relative \( (U, Sp) \)-bordism theory. Note that if we used \( (U, Sp) \) theory instead we could halve \( x_1 \) but not the higher 2-primary Milnor generators; see the discussion of \( (U, fr) \) theory in [2].

2. A conjecture of Ray. We use the preceding observations to establish a conjecture due essentially to Ray [5] concerning the generalised homology (Atiyah-Hirzebruch) spectral sequence

$$E^2_{*,*} = H_*(MU; MSp) \Rightarrow MSp_*(MU).$$

We identify \( E^2_{0,i} \) with \( MSp_i \) and \( E^2_{i,0} \) with \( H_i(MU; \mathbb{Z}) \).

**Theorem 2.1.** The class of \( \mu_n \in E^2_{0,8n-3} \) persists nonzero to \( E^2_{0,8n-3} \) where it is killed by a transgressive differential from a class in \( H_{8n-2}(MU; \mathbb{Z}) \) which is half of the Hurewicz image of a (properly chosen) \( x_{4n-1} \).

**Proof.** Since \( MU_*(MSp) = 0 \) for \( i \) odd it is clear that \( \mu_n \) is killed. Consider the commutative diagram (2.2):

\[
\begin{array}{ccc}
\pi_{8n-2}(MU/MSp) & \xrightarrow{\delta_*} & \pi_{8n-3}(MSp) \\
\uparrow m & & \uparrow m_* \\
MSp_{8n-2}(MU/MSp) & \xrightarrow{\delta_*} & MSp_{8n-3}(MSp) \\
\uparrow i_* & & \uparrow i_* \\
MSp_{8n-2}(MU/S) & \xrightarrow{\delta_*} & MSp_{8n-3}
\end{array}
\]

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Explanation of (2.2) $\text{MU}/S$ is the Thom spectrum of $(U, fr)$ bordism, $i$ is the map of the cofibration sequence

$$ S \xrightarrow{j} MU \xrightarrow{p} \text{MU}/S $$

to the cofibration sequence (1.1) and $\delta: \text{MU}/S \to S^1$ is the cofibre of $\tilde{p}$. The maps $m_*$ arise from the fact that (1.1) is a sequence of spectra with compatible MSp actions and so the $(U, Sp)$ relative bordism sequence is a long exact sequence of $\text{MSp}_*$ modules.

Since $j_*\mu_n = 0$, $\mu_n$ is in the image of $\delta_*$. The assertion that $[\mu_n] \neq 0 \in E_{1,0}^{s-2}$ is equivalent to the statement that $\mu_n$ is not in the image of the composition

$$ MS_{8n-2}([\text{MU}/S]^{(s-3)}) \to MS_{8n-2}(\text{MU}/S) \xrightarrow{\delta_*} MS_{8n-3} $$

(where $[X]^r$ denotes the $r$-skeleton of $X$).

Now suppose that

$$ y_n \in MS_{8n-2}(\text{MU}/S) $$

is in the image of $MS_{8n-2}([\text{MU}/S]^{(s-3)})$ and that $\delta_* y_n = \mu_n$. Then $\partial_* [m_\ast i \ast y_n] = m_\ast i \ast \partial_* y_n = m_\ast i \ast \mu_n = \mu_n$; therefore $m_\ast i \ast y_n \in (\text{MU}/\text{MSp})_{8n-2}$ is a $\bar{\mu}_n$, 'half of a Milnor generator' and can be represented by a map $g_k: S_{8n-2} \to \text{MU}/\text{MSp}$ which we know to be detected in $Z_2$-cohomology by (the anti-image under $p^*$ of) the primitive Chern class; in other words, the image of the map

$$ g_k^*: H_{8n-2}(S_{8n-2}; Z) \to H_{8n-2}(\text{MU}/\text{MSp}; Z) $$

contains an odd integral multiple of the image under $p_\ast$ of an indecomposable element of $H_{8n-2}(\text{MU}; Z)$. But this is inconsistent with the supposition that $g_k^*$ can be factored through $H_{8n-2}(\text{MSp} \wedge \text{MU}/S^{(s-3)})$. (Observe that $m_\ast$ is, in homology, determined by the ordinary product on $H_\ast(\text{MU})$.) So $[\mu_n] \neq 0 \in E_{0,0}^{s-2}$. Finally, the image of

$$ y_n \in H_{8n-2}(\text{MU}; Z) $$

will be $j^{-1} h_\ast m_\ast i \ast y_n = j^{-1} h_\ast \bar{\mu}_n$. We choose $x_{4n-1}$ so that $j_\ast x_{4n-1} = 2\bar{\mu}_n$ and we are finished.

3. Ray’s Hattori-Stong conjecture. In [6] we left open the question of whether $\text{MSp}_{31}$ is 0 or $Z_2$. Here we note that in fact since in the Adams’ spectral sequence for $\text{MSp}$ the map $k_1: E_3^{25}(\text{MSp}) \to E_3^{34}(\text{MSp})$ given by multiplication by $k_1$ is an isomorphism onto, and since $k_1$ and all elements of $E_3^{34}(\text{MSp})$ are permanent cycles it follows that all elements of $E_3^{34}(\text{MSp})$ are permanent cycles; $d_3(v_3^2) \neq 0$ by the results of [3] or [7] and so $\text{MSp}_{31} = Z_2$. This gives us a class in $\text{MSp}_\ast$ which cannot be detected by $KO$-characteristic numbers and so Ray’s ‘mod 2 $\text{MSp}$ Hattori-Stong conjecture’ [4] is false.
Note. S. Kochman has informed me that in certain details the spectral sequence calculations in [6] are in error, e.g. \( d_2v_4 = \gamma_{0,2} + \gamma_{0,1}v_2 \) not \( \gamma_{0,2} \) as claimed. These errors do not affect the validity of the above observations. Kochman also finds the Hattori-Stong conjecture to be false but on other grounds.

We hope to discuss the question of the nonvanishing of three-fold products of torsion elements further in a subsequent note.

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Bibliography


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