PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 3, August 1973

HALVING THE MILNOR MANIFOLDS AND SOME CONJECTURES OF RAY

DAVID M. SEGAL[1]

ABSTRACT. The peculiar properties of the 2^j-2 dimensional generators of unitary bordism (the 2-primary Milnor generators) are related to the 2^j-3 dimensional indecomposable torsion classes of Alexander. This result is then used to confirm a conjecture of Ray concerning the generalised homology spectral sequence for $MSp_*(MU)$. Finally it is noted that Ray's conjecture to the effect that all classes in MSp_* are detectable by KO-characteristic numbers must fail.

1. **Halving the Milnor manifolds.** We consider the cofibration sequence of spectra

$$MSp \xrightarrow{j} MU \xrightarrow{p} MU/MSp; \tag{1.1}$$

MU/MSp will be the Thom spectra for (U, Sp) relative bordism. Recall that $MU_*=Z[x_1, x_2, \cdots, x_i, \cdots]$, $x_i \in MU_{2i}$ and that for $i=2^k-1$ the x_i (the 2-primary Milnor generators) have the peculiarity that, if h_* is the Hurewicz map, then $h_*(x_i)$ is divisible by 2 in $H_{2i}(MU; Z)$ although of course x_i is not divisible in MU_*. We will now shed some light on this peculiar behavior.

J. C. Alexander [1] has defined a series of indecomposable torsion elements $\mu_i \in MSp_{8i-3}$, $2\mu_i=0$. Let σ_1 be the generator of MSp. In all that follows n will be equal to 2^{k-2}. Let $\partial: MU/MSp \to SMSp$ be the cofibre of p. Since $j_*\mu_n=0$, we can choose $\bar{\mu}_n \in (MU/MSp)_{8n-2}$ such that $\partial_*\bar{\mu}_n=\mu_n$. Let

$$f_k: S^{8n-3} \to MSp; \qquad g_k: S^{8n-2} \to MU/MSp$$

classify μ_n, $\bar{\mu}_n$ respectively; $\partial \circ g_k \simeq Sf_k$. Let $U_2 \in H^0(MSp; Z_2)$ be the Thom class and $s^i \in H^i(S^i; Z_2)$ the generator. Then Alexander has shown that f_k is detected by a functional cohomology operation, namely,

$$Sq_{f_k}^{2\Delta(k)}(U_2) = s^{8n-3}$$

Received by the editors October 27, 1972.

AMS (MOS) subject classifications (1970). Primary 57D90, 55B20; Secondary 55H25.

Key words and phrases. Unitary bordism, symplectic bordism, Milnor manifolds, Hattori-Stong conjecture.

[1] This research was supported by the Research Foundation of the City University of New York.

or, suspending once,

$$Sq^{2\Delta(k)}_{Sf_k}(SU_2) = s^{8n-2}$$

But $\partial^*(SU_2)=0$ so

$$g_k^* Sq^{2\Delta(k)}_{\partial}(SU_2) = s^{8n-2}$$

and g_k can be detected by cohomology in the ordinary way. ($Sq^{2\Delta(k)}$ denotes $Sq^{(0,0,0,\cdots,0,2,0,\cdots)}$ (2 in the kth position) in the Milnor basis notation.) Recall the definition of functional operations and compute $Sq^{2\Delta(k)}_{\partial}(SU_2)$:

$$S[Sq^{2\Delta(k)}_{\partial}(SU_2)] = (Sp)^* Sq^{2\Delta(k)}(Sj)^{*-1}U_2.$$

Now $(Sj)^{*-1}(U_2)$ is well defined; it is just (the suspension of) the mod 2 Thom class of MU; call it U_2'. $Sq^{2\Delta(k)}U_2'$ is the primitive class in the coalgebra $H^{8n-2}(MU; Z_2)$. Thus taking normal Chern numbers on $(MU/MSp)_*$ we have that $s_{(4n-1)}(c)[\bar{\mu}_n]$ is odd. Since $p_*(2\bar{\mu})=2\mu_n=0$ we can pull $2\bar{\mu}_n$ back to MU_{8n-2} and the pull back can be chosen as x_{4n-1}, a 2-primary Milnor generator. In other words, the 2-primary Milnor generators (properly chosen) can be halved by being regarded as classes in the relative (U, Sp)-bordism theory. Note that if we used (U, fr) theory instead we could halve x_1 but not the higher 2-primary Milnor generators; see the discussion of (U, fr) theory in [2].

2. **A conjecture of Ray.** We use the preceding observations to establish a conjecture due essentially to Ray [5] concerning the generalised homology (Atiyah-Hirzebruch) spectral sequence

$$E^2_{**} = H_*(MU; MSp) \Rightarrow MSp_*(MU).$$

We identify $E^2_{0,i}$ with MSp_i and $E^2_{i,0}$ with $H_i(MU; Z)$.

THEOREM 2.1. *The class of $\mu_n \in E^2_{0,8n-3}$ persists nonzero to $E^{8n-2}_{0,8n-3}$ where it is killed by a transgressive differential from a class in $H_{8n-2}(MU; Z)$ which is half of the Hurewicz image of a (properly chosen) x_{4n-1}.*

PROOF. Since $MU_i(MSp)=0$ for i odd it is clear that μ_n is killed. Consider the commutative diagram (2.2):

$$\begin{tikzcd}
\pi_{8n-2}(MU/MSp) \arrow[r, "\partial_*"] & \pi_{8n-3}(MSp) \\
MSp_{8n-2}(MU/MSp) \arrow[r, "\partial_*"] \arrow[u, "m_*"] & MSp_{8n-3}(MSp) \arrow[u, "m_*"] \\
MSp_{8n-2}(MU/S) \arrow[r, "\bar{\partial}_*"] \arrow[u, "i_*"] & MSp_{8n-3} \arrow[u, "i_*"]
\end{tikzcd} \tag{2.2}$$

Explanation of (2.2) MU/S is the Thom spectrum of (U,fr) bordism, i is the map of the cofibration sequence

$$S \xrightarrow{\bar{j}} MU \xrightarrow{\bar{p}} MU/S$$

to the cofibration sequence (1.1) and $\bar{\partial}: MU/S \to S^1$ is the cofibre of $\bar{p}$. The maps m_* arise from the fact that (1.1) is a sequence of spectra with compatible MSp actions and so the (U, Sp) relative bordism sequence is a long exact sequence of MSp_* modules.

Since $\bar{j}_*\mu_n=0$, μ_n is in the image of $\bar{\partial}_*$. The assertion that $[\mu_n]\neq 0 \in E^{8n-2}_{0,8n-3}$ is equivalent to the statement that μ_n is not in the image of the composition

$$MSp_{8n-2}([MU/S]^{(8n-3)}) \longrightarrow MSp_{8n-2}(MU/S) \xrightarrow{\bar{\partial}_*} MSp_{8n-3}$$

(where $[X]^{(r)}$ denotes the r-skeleton of X).

Now suppose that

$$y_n \in MSp_{8n-2}(MU/S)$$

is in the image of $MSp_{8n-2}([MU/S]^{(8n-3)})$ and that $\bar{\partial}_* y_n=\mu_n$. Then $\partial_*[m_*i_*y_n]=m_*i_*\bar{\partial}_*y_n=m_*i_*\mu_n=\mu_n$; therefore $m_*i_*y_n \in (MU/MSp)_{8n-2}$ is a $\bar{\mu}_n$, 'half of a Milnor generator' and can be represented by a map $g_k: S^{8n-2}\to MU/MSp$ which we know to be detected in Z_2-cohomology by (the anti-image under p^* of) the primitive Chern class; in other words, the image of the map

$$g_{k*}: H_{8n-2}(S^{8n-2}; Z) \to H_{8n-2}(MU/MSp; Z)$$

contains an odd integral multiple of the image under p_* of an indecomposable element of $H_{8n-2}(MU; Z)$. But this is inconsistent with the supposition that g_{k*} can be factored through $H_{8n-2}(MSp\wedge MU/S^{(8n-3)})$. (Observe that m_* is, in homology, determined by the ordinary product on $H_*(MU)$.) So $[\mu_n]\neq 0 \in E^{8n-2}_{0,8n-3}$. Finally, the image of

$$y_n \in H_{8n-2}(MU; Z)$$

will be $j_*^{-1}h_*m_*i_*y_n=j_*^{-1}h_*\bar{\mu}_n$. We choose x_{4n-1} so that $j_*x_{4n-1}=2\bar{\mu}_n$ and we are finished.

3. **Ray's Hattori-Stong conjecture.** In [6] we left open the question of whether MSp_{31} is 0 or Z_2. Here we note that in fact since in the Adams' spectral sequence for MSp the map $k_1: E_3^{2,28}(MSp)\to E_3^{3,34}(MSp)$ given by multiplication by k_1 is an isomorphism onto, and since k_1 and all elements of $E_3^{2,28}(MSp)$ are permanent cycles it follows that all elements of $E_3^{3,34}(MSp)$ are permanent cycles; $d_3(v_4^2)\neq 0$ by the results of [3] or [7] and so $MSp_{31}= Z_2$. This gives us a class in MSp_* which can not be detected by KO-characteristic numbers and so Ray's 'mod 2 MSp Hattori-Stong conjecture' [4] is false.

Note. S. Kochman has informed me that in certain details the spectral sequence calculations in [6] are in error, e.g. $d_2 v_4 = \gamma_{0,2} + \gamma_{0,1} v_2$ not $\gamma_{0,2}$ as claimed. These errors do not affect the validity of the above observations. Kochman also finds the Hattori-Stong conjecture to be false but on other grounds.

We hope to discuss the question of the nonvanishing of three-fold products of torsion elements further in a subsequent note.

4. **Acknowledgements.** We wish to thank J. C. Alexander, N. Ray and S. Kochman for informing us of their various studies of MSp_*. We also wish to point out that Alexander has obtained the results of §1 by other methods; his results will appear in Topology.

Bibliography

1. J. C. Alexander, *A family of indecomposable symplectic manifolds*, Amer. J. Math. **94** (1972), 699–710.

2. P. E. Conner and E. E. Floyd, *The relation of cobordism to K-theories*, Lecture Notes in Math., no. 28, Springer-Verlag, Berlin and New York, 1966. MR **35** #7344.

3. E. E. Floyd, *Stiefel-Whitney numbers of quaternionic and related manifolds*, Trans. Amer. Math. Soc. **155** (1971), 77–94. MR **42** #8509.

4. N. Ray, *A note on the symplectic bordism ring*, Bull. London Math. Soc. **3** (1971), 159–162.

5. ———, *The symplectic bordism ring*, Proc. Cambridge Philos. Soc. **71** (1972), 271–282.

6. D. M. Segal, *On the symplectic cobordism ring*, Comment. Math. Helv. **45** (1970), 159–169. MR **41** #7699.

7. ———, *Divisibility conditions on characteristic numbers of stably symplectic manifolds*, Proc. Amer. Math. Soc. **27** (1971), 411–415. MR **42** #5282.

Department of Mathematics, City College of the City University of New York, New York, New York 10031