ERGODIC PROPERTIES OF BOUNDED $L_1$-OPERATORS

RYOTARO SATO

Abstract. Individual ergodic theorems for bounded $L_1$-operators are proved in §1, and the problem of existence of positive invariant functions for positive $L_1$-operators is considered in §2. A decomposition theorem similar to that of Sucheston [12] is proved in the last section.

1. Individual ergodic theorems. Let $(X, \mathcal{M}, m)$ be a $\sigma$-finite measure space and $L_p(X) = L_p(X, \mathcal{M}, m), 1 \leq p \leq \infty$, the usual (complex) Banach spaces. If $A \in \mathcal{M}$ then $1_A$ is the indicator function of $A$ and $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish a.e. on $X - A$. Let $T$ be a bounded linear operator on $L_1(X)$ and $\tau$ its linear modulus [2]. Thus $\tau$ is a positive linear operator on $L_1(X)$ such that $\|\tau\|_1 = \|\tau\|_1$ and $\tau g = \sup\{|Tf|; f \in L_1(X)$ and $|f| \leq g\}$ for any $0 \leq g \in L_1(X)$. The adjoint of $T$ is denoted by $T^*$. Throughout this section it will be assumed that there exists a strictly positive function $s$ in $L_\infty(X)$ such that

$$\tau^* s \leq s \quad \text{a.e.}$$

Clearly if $T$ is a contraction then $\tau^* 1 \leq 1$ a.e. Let $a_{n,k}$ $(n, k = 0, 1, \cdots)$ be a matrix of numbers such that

$$\lim_{n} \sum_{k=0}^{\infty} a_{n,k} = 1,$$

(2)

$$\lim_{n'} \sum_{k=0}^{\infty} a_{n',k} b_{k+1} = b$$

whenever $b_0, b_1, \cdots$ is a bounded sequence of numbers for which

$$\lim_{n'} \sum_{k=0}^{\infty} a_{n',k} b_k = b$$

exists and is finite, where $(n')$ is a subsequence of $(n)$. Let $w_1, w_2, \cdots$ be a sequence of nonnegative numbers whose sum is one, and let $u_0, u_1, \cdots$ be the sequence defined by $u_0 = 1$ and $u_n = w_n u_0 + \cdots + w_1 u_{n-1}$ for $n \geq 1$. In this section, under these conditions, we shall prove the following theorems.
Theorem 1. If \( p_0, p_1, \ldots \) is a sequence of nonnegative measurable functions on \( X \) with \( |Tg| \leq P_{n+1} \) a.e. whenever \( g \in L_1(X) \) and \( |g| \leq p_n \) a.e. then for any \( f \in L_1(X) \) the limit

\[
(*) \quad \lim_n \left( \frac{\sum_{k=0}^{n} u_k T^k f(x)}{\sum_{k=0}^{n} u_k p_k(x)} \right)
\]

exists and is finite a.e. on the set \( \{ x \in X ; \sum_{k=0}^{\infty} u_k p_k(x) > 0 \} \).

Theorem 2. Suppose there exists a strictly positive function \( h \) in \( L_1(X) \) such that

(i) \( \sum_{k=0}^{n} a_{n,k} T^k h \) exists in the weak topology for any \( n \), and

(ii) the set \( \{ \sum_{k=0}^{n} a_{n,k} T^k h ; n \geq 0 \} \) is weakly sequentially compact in \( L_1(X) \).

Then for any \( f \in L_1(X) \) the limit

\[
(**) \quad \lim_n \left( \frac{\sum_{k=0}^{n} u_k T^k f(x)}{\sum_{k=0}^{n} u_k} \right)
\]

exists and is finite a.e.

Proof of Theorem 1. For \( sf \in L_1(X) \), where \( f \in L_1(X) \), define

\[
V_T(sf) = sTf \quad \text{and} \quad V_r(sf) = srf.
\]

Since \( \{sf ; f \in L_1(X)\} \) is a dense subspace of \( L_1(X) \) and \( \|V_T(sf)\|_1 \leq \|V_r(sf)\|_1 = \|sTf\|_1 = \|s\|_1 \), \( V_T \) and \( V_r \) may be considered to be linear contractions on \( L_1(X) \). An easy argument shows that \( V_r \), coincides with the linear modulus of \( V_T \). Let \( g \in L_1(X) \) and \( |g| \leq sp_n \) a.e., and choose an increasing sequence \( g_1, g_2, \ldots \) of nonnegative integrable functions on \( X \) such that \( \lim_n s g_n = |g| \) a.e. Then \( |V_T g| \leq V_r |g| = \lim_n s r g_n \leq s p_{n+1} \) a.e., and hence the ergodic theorem of [9] completes the proof of Theorem 1.

Proof of Theorem 2. Let \( g \in L_1(X) \) and \( (n') \) a subsequence of \( (n) \) such that \( \sum_{k=0}^{\infty} a_{n',k} T^k h \) converges weakly to \( g \). Then it follows from a slight modification of an argument of [10] that

\[
(5) \quad \lim_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} V_r^k (sh) - sg \right\|_1 = 0,
\]

and that \( sg > 0 \) a.e. on \( C \) and \( sg = 0 \) a.e. on \( D \), where \( C \) and \( D \) denote the conservative and dissipative parts [1] of \( V_r \), respectively. Hence Theorem 1 completes the proof of Theorem 2.

2. Invariant functions. In this section we shall assume that \( (X, M, m) \) is a probability space and \( T \) is a positive linear operator on \( L_1(X) \) such that there exists a strictly positive function \( s \) in \( L_\infty(X) \) with \( T^* s \leq s \) a.e.
The operator $T$ is called conservative if $\sum_{k=0}^{\infty} T^k f(x) = \infty$ a.e. for any strictly positive function $f \in L_1(X)$. A measurable set $A$ is called closed if $f \in L_1(A)$ implies $Tf \in L_1(A)$. The purpose of this section is to prove the following theorems.

**Theorem 3.** If $T$ is conservative and satisfies the condition of Theorem 2 then there exists a strictly positive function $g \in L_1(X)$ with $Tg = g$ and hence if, in addition,

\[ \sup_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k \right\|_1 < \infty \]

then the mean ergodic theorem holds for $T$; i.e., for any $f \in L_1(X)$ the sequence $(1/n) \sum_{k=0}^{n-1} T^k f$ converges in the norm topology.

**Corollary 1 (cf. Fong [5, Theorem 3]).** If $T$ satisfies (6), then a necessary and sufficient condition that $T$ have a strictly positive invariant function in $L_1(X)$ is that $T$ be conservative and for any $A \in \mathcal{M}$ the limit

\[ \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} T^{*k}1_A \, dm \]

exist.

**Theorem 4.** If $T$ satisfies (6), then the following conditions are equivalent.

(i) $A \in \mathcal{M}$ and $m(A) > 0$ imply $\inf_{n} \int T^{*n}1_A \, dm > 0$.

(ii) $A \in \mathcal{M}$ and $m(A) > 0$ imply

\[ \lim_{n} \left( \sup_{j} \frac{1}{n} \sum_{k=0}^{n-1} T^{*k+j}1_A \, dm \right) > 0. \]

**Theorem 5.** If $T$ satisfies (6), then the space $X$ is the disjoint union of two uniquely determined measurable sets $P$ and $N$ such that

(a) $P$ is closed,
(b) there exists an $h \in L_1(P)$ with $h > 0$ a.e. on $P$ and $Th = h$,
(c) for any $f \in L_1(X)$ the limit

\[ \tilde{f}(x) = \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} T^k f(x) \]

exists and is finite a.e., $\tilde{f} \in L_1(P)$ and $T\tilde{f} = \tilde{f}$ a.e.; moreover we have

\[ \lim_{n} \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k f - \tilde{f} \right\|_1 = 0, \]
(d) if $N = X - P$ then $N$ is a union of countably many sets $A_i \in \mathcal{M}$ with
\[ \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \int_{A_i} T^k f \, dm = 0 \]
for any $0 \leq f \in L_1(X)$.

Theorem 4 is a generalization of results obtained by Neveu [7] (see also [8]), Dean and Sucheston [3] and Fong [5], and Theorem 5 is a generalization of results obtained by Krengel [6] and Fong [5].

**Proof of Theorem 3.** The first half of the theorem is direct from the argument in the proof of Theorem 2, and the second half follows from the mean ergodic theorem (cf. [4, Theorem VIII.5.1]).

**Proof of Theorem 4.** For the purpose of proof we introduce a third condition:

(iii) $A \in \mathcal{M}$ and $m(A) > 0$ imply
\[ \liminf_n \frac{1}{n} \sum_{k=0}^{n-1} \int_{A} T^k u \, dm > 0. \]

The proof follows the scheme $(0) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (0)$. The implication $(i) \Rightarrow (iii)$ is obvious. The following two implications $(0) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ follow from the same arguments as in [5, p. 80]. Thus we prove here only the implication (iii) $\Rightarrow (0)$.

Let $L$ be a Banach limit and define a positive linear functional $\varphi$ on $L_\infty(X)$ by the relation
\[ \varphi(u) = L(\frac{1}{n} \sum_{k=0}^{n-1} \int_{A} T^k u \, dm), \quad u \in L_\infty(X). \]

If we denote by $T^{**}$ the adjoint of $T^*$ then, for any $0 \leq u \in L_\infty(X)$,
\[ (T^{**} \varphi - \varphi)u = \varphi(T^*u - u) = L((1/n) \int (T^*u - u) \, dm) \geq L((1/n) \int T^*u \, dm) \geq 0, \]
and hence $T^{**} \varphi - \varphi \geq 0$. Thus if we let $\varphi_n = (1/n) \sum_{k=0}^{n-1} T^{**} \varphi_k$ then
\[ 0 \leq \varphi_n \leq \varphi_1 \leq \varphi_2 \leq \cdots \]
and the $\|\varphi_n\|$ are bounded, whence there exists a positive linear functional $\varphi_\infty$ on $L_\infty(X)$ such that $\lim_n \|\varphi_n - \varphi_\infty\| = 0$. It is now easy to see that $T^{**} \varphi_\infty = \varphi_\infty$. Set $\mu(A) = \varphi_\infty(1_A)$ for $A \in \mathcal{M}$. Then $\mu$ is a finitely additive measure on $\mathcal{M}$ vanishing on sets of $m$ measure zero. Let $\mu = \mu_m + \mu_c$ be the unique decomposition of $\mu$, where $\mu_m \geq 0$ is a countably additive measure on $\mathcal{M}$ and where $\mu_c \geq 0$ is a finitely additive measure on $\mathcal{M}$ such that if $\lambda \geq 0$ is a countably additive measure on $\mathcal{M}$ with $\lambda \leq \mu_c$ then $\lambda = 0$ [11, Theorem 3]. Then $T^{**} \mu_m \leq T^{**} \mu = \mu = \mu_m + \mu_c$ and hence $T^{**} \mu_m = \mu_m$ from which it follows easily that $T^{**} \mu_m \leq \mu_m$.

We next show that $\mu_m$ is equivalent to $m$. Assume the contrary: there exists a set $E \in \mathcal{M}$ with $\mu_m(E) = 0$ and $m(E) > 0$. Then Theorem 4 of [11] implies that there exists a set $A \in \mathcal{M}$ with $A \subset E$, $\mu_c(A) = m(A) = 0$ and
$m(A) > 0$. But this is impossible because, by (iii),

$$
\mu(A) \geq \varphi_{\infty}(1_A) \geq \varphi(1_A) = L \left( \frac{1}{n} \sum_{k=0}^{n-1} T^{*k}1_A \, dm \right)
$$

$$
\geq \lim \inf \frac{1}{n} \sum_{k=0}^{n-1} \int T^{*k}1_A \, dm > 0.
$$

To complete the proof it is now sufficient to show that $T^{**}\mu_m = \mu_m$. But to see this, since $T^{**}\mu_m \leq \mu_m$, it suffices to show that $T^*s = s$. If this fails to hold then there exists a set $A \in \mathcal{M}$ with $m(A) > 0$ and a positive constant $c$ such that $s - T^*s \geq c1_A$, and hence we have

$$
\lim \inf \frac{1}{n} \sum_{k=0}^{n-1} \int cT^{*k}1_A \, dm \leq \lim \inf \frac{1}{n} \sum_{k=0}^{n-1} \int T^{*k}(s - T^*s) \, dm
$$

$$
= \lim \inf \frac{1}{n} \int (s - T^*s) \, dm
$$

$$
\leq \lim \frac{1}{n} \int s \, dm = 0,
$$

which contradicts (iii). This completes the proof of Theorem 4.

**Proof of Theorem 5.** An argument similar to that of [5, Proposition 2] is sufficient, and we omit the details.

**Remark 1.** It may be readily seen that if $T$ has a strictly positive invariant function $f_0 \in L_1(X)$ then the class $\mathcal{S}$ of all closed sets forms a $\sigma$-subfield of $\mathcal{M}$ (cf. [1]). Thus if $f \in L_1(X)$, we shall denote by $E\{f|\mathcal{S}\}$ the conditional expectation of $f$ with respect to $\mathcal{S}$. Applying the Chacon identification theorem [1], we have the following results.

(a) If $f \in L_1(X)$ then

$$
\lim \frac{1}{n} \sum_{k=0}^{n-1} T^k f = f_0 \frac{E\{sf_0|\mathcal{S}\}}{E\{s|\mathcal{S}\}} \quad \text{a.e.}
$$

(b) If $u \in L_\infty(X)$ then

$$
\lim \frac{1}{n} \sum_{k=0}^{n-1} T^{*k} u = s \frac{E\{uf_0|\mathcal{S}\}}{E\{s|\mathcal{S}\}} \quad \text{a.e.}
$$

3. **Decomposition theorem.** In this section we shall prove the following

**Theorem 6 (cf. Sucheston [12, Theorem 1]).** If $T$ is a positive linear operator on $L_1(X)$ satisfying (6), then the space $X$ uniquely decomposes into two measurable sets $Y$ and $Z$ such that

(i) $f \in L_1(Z)$ implies $Tf \in L_1(Z)$,

(ii) if $f \in L_1(Z)$ then $\lim_n \| (1/n) \sum_{k=0}^{n-1} T^k f \|_1 = 0$,
(iii) there exists a nonnegative function $s$ in $L_\infty(Y)$ with $s>0$ a.e. on $Y$ and $T^*s=s$.

**Proof.** If we let $u=\limsup_n (1/n) \sum_{k=0}^{n-1} T^*k$, then an easy calculation shows that $u \in L_\infty(X)$ and $T^*u \geq u$ a.e. Next if we let $s=\lim_n (1/n) \sum_{k=0}^{n-1} T^*ku$, then it follows that $0 \leq s \in L_\infty(X)$ and $T^*s=s$. Put $Y=\{x \in X; s(x)>0\}$ and $Z=X-Y$. If $0 \leq f \in L_1(Z)$ then

$$
\lim_n \int \left( \frac{1}{n} \sum_{k=0}^{n-1} T^kf \right) dm = \lim_n \int f \left( \frac{1}{n} \sum_{k=0}^{n-1} T^*k \right) dm \\
\leq \int fu dm \leq \int fs dm = 0.
$$

Thus (ii) follows. (i) is clear. The proof is complete.

**Remark 2.** By Theorem 1, if $f \in L_1(X)$ and $0 \leq g \in L_1(X)$ then the limit

$$
\lim_n \left( \sum_{k=0}^{n} T^k f(x) \right) / \left( \sum_{k=0}^{n} T^k g(x) \right)
$$

exists and is finite a.e. on $Y \cap \{x \in X; \sum_{k=0}^{\infty} T^k g(x)>0\}$. But in general this does not hold on $Z \cap \{x \in X; \sum_{k=0}^{\infty} T^k g(x)>0\}$ (see Fong [5, p. 77]).

**Corollary 2.** Let $T$ be a positive linear operator on $L_1(X)$ satisfying (6), and suppose that $\limsup_n (1/n) \sum_{k=0}^{n-1} T^k \|g\|_1>0$ for any $0 \leq g \in L_1(X)$ with $\|g\|_1>0$. Then there exists a strictly positive function $s$ in $L_\infty(X)$ with $T^*s=s$.

**Bibliography**


Department of Mathematics, Josai University, Sakado, Saitama 350-02, Japan