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A NEW PROOF OF THE CONSTRUCTION THEOREM FOR STONE ALGEBRAS

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ABSTRACT. A simple proof is given of Chen's and Grätzer's theorem, which gives a method to construct a Stone algebra from a Boolean algebra and a distributive lattice with 1 by certain connective conditions between the two given lattices.

C. C. Chen and G. Grätzer [1] proved originally the Construction Theorem for Stone algebras. In [3] we proved by different method the Construction Theorem for a larger class of structures with pseudocomplementation than the class of Stone algebras. Modifying the method from [1] we have proved in [4] the Construction Theorem for distributive lattices with pseudocomplementation. The proofs of all mentioned theorems are rather complicated. G. Grätzer in his book [2] set a task (Problem 55): "Find a direct (less-computational) proof of the Construction Theorem for Stone algebras".

In this note we shall give an answer to this problem. It will be a simpler proof of the Construction Theorem.

Preliminaries. A universal algebra $\langle L;\ \cup,\ \cap,\ {}^*,\ 0,\ 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ is called a *distributive p-algebra* iff $\langle L;\ \cup,\ \cap,\ 0,\ 1\rangle$ is a bounded distributive lattice such that for every $a \in L$ the element a^* is the *pseudocomplement* of a, i.e. $x \leqq a^*$ iff $a \cap x = 0$. A distributive p-algebra satisfying the Stone identity $x^* \cup x^{**} = 1$ is called a *Stone algebra*. The standard results on Stone algebras may be found in [2].

For a Stone algebra L define the set $B(L) = \{x \in L : x = x^{**}\}$ of *closed* elements. The partial ordering of L partially orders $B(L)$ and turns the latter into a Boolean algebra $\langle B(L);\ \cup,\ \cap,\ {}^*,\ 0,\ 1\rangle$. Another significant subset of a Stone algebra L is the set of *dense* elements $D(L) = \{x \in L : x^* = 0\}$. $D(L)$ is a filter (dual ideal) in L.

Let $F(D)$ denote the set of all filters of D ordered by the set inclusion. $F(D(L))$, for a Stone algebra L, is a distributive lattice. Finally define a

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mapping $\varphi(L): B(L) \to F(D(L))$ by

$$\varphi(L): a \to \{x \in D(L): x \geqq a^*\}.$$

A $\{0, 1\}$-*homomorphism* (of a bounded lattice into another one) is a homomorphism taking zero into zero and unit into unit.

The relationship between $B(L)$, $D(L)$ and $\varphi(L)$ is expressed in the following.

THEOREM ([**1**], [**2**]). *Let L be a Stone algebra. Then $B(L)$ is a Boolean algebra, $D(L)$ is a distributive lattice with* 1, *and $\varphi(L)$ is a* $\{0, 1\}$-*homomorphism of $B(L)$ into $F(D(L))$. The triple $\langle B(L), D(L), \varphi(L)\rangle$ characterizes L up to isomorphism.*

The construction. Now we can give a new and simpler proof of the following.

CONSTRUCTION THEOREM ([**1**], [**2**]). *Given a Boolean algebra* $\langle B; \cup, \cap, ', 0, 1\rangle$, *a distributive lattice D with* 1, *and a* $\{0, 1\}$-*homomorphism* $\varphi: B \to F(D)$, *there exists a Stone algebra L whose triple is* $\langle B, D, \varphi\rangle$.

PROOF. Let $[t)=\{x \in D: t \in D$ and $x \geqq t\}$ for each $t \in D$. Put $a\theta=a'\varphi$ for $a \in B$. Then $0\theta=D$, $1\theta=[1)$, $(a\cap b)\theta=a\theta\cup b\theta$ and $(a\cup b)\theta=a\theta\cap b\theta$. Let $\langle F_d(D); \vee, \wedge\rangle$ denote the dual lattice to the distributive lattice $\langle F(D); \cup, \cap\rangle$ of all filters of D, i.e. $J\vee K=J\cap K$, $J\wedge K=J\cup K$ for J, $K \in F(D)$.

Since $a\varphi$ ($a \in B$) belongs to the centre of $F(D)$, for every $x \in D$, there exists $t \in D$ such that $[t)=a\varphi\cap[x)$. Denote the element t by $x\rho_a$.

Set

$$L = \{\langle a, a\theta \cup [x)\rangle : a \in B, x \in D\}. \tag{1}$$

L is a subset of the direct product $B \times F_d(D)$. We show first that L is a sublattice of $B \times F_d(D)$. Let $\langle a, a\theta\cup[x)\rangle$, $\langle b, b\theta\cup[y)\rangle \in L$. Then we have in the lattice $B \times F_d(D)$

$$\begin{aligned} &\langle a, a\theta \cup [x)\rangle \cap \langle b, b\theta \cup [y)\rangle \\ &\quad = \langle a \cap b, a\theta \cup b\theta \cup [x) \cup [y)\rangle = \langle a \cap b, (a \cap b)\theta \cup [x \cap y)\rangle \end{aligned} \tag{2}$$

and

$$\langle a, a\theta \cup [x)\rangle \cup \langle b, b\theta \cup [y)\rangle = \langle a \cup b, (a\theta \cup [x)) \cap (b\theta \cup [y))\rangle. \tag{3}$$

By distributivity of $F(D)$ we obtain

$$\begin{aligned} &(a\theta \cup [x)) \cap (b\theta \cup [y)) \\ &\quad = (a\theta \cap b\theta) \cup (a\theta \cap [y)) \cup ([x) \cap b\theta) \cup ([x) \cap [y)) \\ &\quad = (a \cup b)\theta \cup [y\rho_{a'} \cap x\rho_{b'} \cap (x \cup y)). \end{aligned} \tag{4}$$

(2)–(4) implies that L is a sublattice of $B\times F_d(D)$. Hence L is distributive. It is easy to see that

(5) $\langle a, a\theta \cup [x)\rangle \leqq \langle b, b\theta \cup [y)\rangle$ iff $a \leqq b$ and $a\theta \cup [x) \supseteq b\theta \cup [y)$

in L. L is bounded and

$$\langle 0, D\rangle \leqq \langle a, a\theta \cup [x)\rangle \leqq \langle 1, [1)\rangle \tag{6}$$

holds.

L is pseudocomplemented and

$$\langle a, a\theta \cup [x)\rangle^* = \langle a', a'\theta\rangle \tag{7}$$

is true. Really,

$$\langle a, a\theta \cup [x)\rangle \cap \langle a', a'\theta\rangle = \langle 0, a\theta \cup a'\theta \cup [x)\rangle = \langle 0, D\rangle.$$

If $\langle a, a\theta\cup[x)\rangle\cap\langle b, b\theta\cup[y)\rangle=\langle 0, D\rangle$ then $b\leqq a'$. The latter implies $b\theta\supseteq a'\theta$. Therefore, by (5),

$$\langle b, b\theta \cup [y)\rangle \leqq \langle a', a'\theta\rangle.$$

Thus (7) is proved.

Now it is easy to show that L is a Stone algebra.

(7) implies

$$B(L) = \{\langle a, a\theta\rangle;\, a \in B\} \quad \text{and} \tag{8}$$

$$D(L) = \{\langle 1, [x)\rangle;\, x \in D\}. \tag{9}$$

Identifying $a\in B$ with $\langle a, a\theta\rangle$ and $d\in D$ with $\langle 1, [d)\rangle$ we can verify $B(L)=B$ and $D(L)=D$. Finally we prove $\varphi(L)=\varphi$. By (5),

$$\langle 1, [x)\rangle \geqq \langle a', a'\theta\rangle \quad \text{iff } [x) \subseteq a'\theta = a\varphi.$$

Hence $x\in D$ and $x\geqq a^*$ in L iff $x\in a\varphi$. Thus $\varphi(L)=\varphi$ and the theorem is proved.

REMARK 1. Another simplification can be obtained if D is supposed to be bounded. Let s be the smallest element of D. It is known that $a\varphi$ ($a\in B$) is a principal filter of D, i.e. there exists $t\in D$ such that $a\varphi=[t)$. Put $[n_a)=a\theta$. Then L can be defined in the following way.

$$L = \{\langle a, d\rangle;\, s \leqq d \leqq n_a\}. \tag{1'}$$

L is a sublattice of the direct product $B\times D$ and the relations (2), (3), (5)–(7) are changed into

(2′) $$\langle a, d\rangle \cap \langle b, e\rangle = \langle a \cap b, d \cap e\rangle;$$

(3′) $$\langle a, d\rangle \cup \langle b, e\rangle = \langle a \cup b, d \cup e\rangle;$$

(5′) $$\langle a, d\rangle \leqq \langle b, e\rangle \quad \text{iff } a \leqq b \text{ and } d \leqq e;$$

(6′) $$\langle 0, s\rangle \leqq \langle a, d\rangle \leqq \langle 1, 1\rangle;$$

(7′) $$\langle a, d\rangle^* = \langle a', n_{a'}\rangle.$$

REMARK 2. We can compare our construction with that given in [**1**]. More precisely, let $\langle B, D, \varphi\rangle$ be a triple described in the Construction Theorem. Let L denote the Stone algebra constructed to $\langle B, D, \varphi\rangle$ while L_1 denotes the Stone algebra corresponding to the same triple in [**1**]. We recall that

$$L_1 = \{\langle x, a\rangle; a \in B, x \in a\varphi\}$$

and

$$\langle x, a\rangle \leqq \langle y, b\rangle \quad \text{iff } a \leqq b,\ x \leqq y\rho_a.$$

Since $(a\theta \cup [x)) \cap a\varphi = [x\rho_a)$, an easy calculation shows that the mapping $\langle a, a\theta \cup [x)\rangle \to \langle x\rho_a, a\rangle$ establishes an isomorphism of L onto L_1.

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