PONTRYAGIN CLASSES OF VECTOR BUNDLES OVER BSp(n)
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Abstract. Let X be a finite skeleton of the classifying space of BSp(n), γ₀→BSp(n), the classifying bundle for Sp(n) vector bundles and γ→X the restriction of γ₀ over X. If ξ→X is another Sp(n) vector bundle, the Pontryagin classes pᵢ(ξ) must be congruent to dᵢ pᵢ(γ) modulo certain odd primes. Equality obtains if ξ is the restriction over X of a γ₀→BSp(n). In particular, Sp(m) vector bundles θ over BSp(n) have p(θ)=1 if m<n.

1. Notation. Let X denote a skeleton of BSp(n), the classifying space for quaternionic n dimensional vector bundles over compact spaces. Let Pᵢ(γ)=xᵢ e Ωⁿᵢ Sp(X) denote the symplectic cobordism Pontryagin classes of the classifying bundle γ₀→BSp(n), pulled back to γ→X by the inclusion. Just as in [1, (8.1)], it can be shown that 1, x₁, x₂, · · · , xₙ generate Ωⁿ Sp(BSp(n)) as a free Ωⁿ Sp-module. If α=ix₁, i₂, · · · , iₙ is a partition of q, let xₐ e Ωⁿ Sp(X) denote the monomial xᵢ₁xᵢ₂ · · · xᵢₙ. Let n(q) denote the set of all partitions of q.

Suppose f is an Sp(n) vector bundle over X. Then

(1.1) Pᵢ(ξ) = ∑ₓₐ∈n(q) dₓₐxₐ + terms having coefficients in Ωⁿ Sp Xₐ, i > 0.

Since the Hurewicz transformation μ:Ωⁿ Sp(X)→H*(X; Z) preserves characteristic classes, the summation term in (1.1) gives the usual Pontryagin classes of f, pᵢ(f), in terms of the generators of H*(X; Z), pᵢ(γ).

Recall (e.g. from [3, p. 179]) that Ωⁿ Sp has 2p−2 dimensional ring generators [Mᵖ⁻¹] whose tangential characteristic numbers, sₜ(p−1)/₂(p(τ(M))), are ±2ʰᵖ, p an odd prime, h some nonnegative integer. There is a relationship to the Landweber cohomology operations [2, §3 and §9],

Lₜ[p−1]/₂([Mᵖ⁻¹]) = −sₜ[p−1]/₂(p(τ(M))).

In the following section, let p denote an odd prime such that 2n−2 divides p−1. Let congruences between integers be modulo p unless otherwise indicated. The set of elements in Ωⁿ Sp(X) of the form αxₐ, where

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\( \alpha \in \Omega^p_2 \) is such that either \( i > 0 \) or \( \alpha \in \rho \Omega^p_2 \), generates an ideal \( H \). In what follows let congruences between elements of \( \Omega^*_p(X) \) be understood to be modulo \( H \). For convenience the coefficients occurring in [4, Example 4, p. 82] will be denoted \( A(\lambda_1, \lambda_2, \ldots, \lambda_n) \). If the sum of \( r \) powers \( s_r = y_1^r + y_2^r + \cdots + y_n^r \) is expressed in terms of the elementary symmetric functions \( s_1, s_2, \ldots, s_n \) then

\[
s_r(\sigma_1, \ldots, \sigma_n) = \sum A(\lambda_1, \ldots, \lambda_n) \sigma_1^{\lambda_1}(-\sigma_2)^{\lambda_2} \cdots (\pm \sigma_n)^{\lambda_n}.
\]

\( s_r(\xi) \) will denote \( s_r \) with \( \sigma_q \) replacing \( P_q(\xi) \). Finally, suppose dimension \( X \) is greater than \( 2(p-1+2n) \).

2. Restrictions mod \( p \). We apply the Landweber operation, \( \mathcal{L}_{(p-1)/2} \), to (1.1) and use the independence of \( \{x_\omega | \omega \in \pi(q), 1 \leq q \leq n \} \).

Consider \( \mathcal{L}_{(p-1)/2}(P_q(\xi)) = \sum d_\omega \mathcal{L}_{(p-1)/2}(x_\omega) \) or:

\[
\sum_{t=1}^q (-1)^{t+1} P_{q-t}(\xi) s_{(p-1)/2+t}(\xi) \\
\equiv \sum_{\omega \in \pi(q)} d_\omega \sum_{a \in \omega} (-1)^{t+1} x_a s_{(p-1)/2+t}(\gamma).
\]

(2.1)

\[
d_{n-1}^{(p-1)/(2n-2)} = d_1.
\]

(2.2)

\[
d_{n-1}^{(p-1)/(2n-2)-j} \equiv d_j \quad \text{if} \quad d_{n-1} \neq 0.
\]

(2.3)

Proof. In (2.1) compare the coefficients of \( x_{n-1}^{(p-1)/(2n-2)+1} \) for \( q = n-1 \). On the left only \( (-1)^n S_{(p-1)/2+1}(\xi) \) may have a monomial term \( x_{n-1}^{(p-1)/(2n-2)+1} \), as for the right of (2.1), only \( d_{n-1}(-1)^n S_{(p-1)/2+n-1}(\gamma) \) may. Thus

\[
A(0, \ldots, 0, (p-1)/(2n-2) + 1, 0)(d_{n-1})^{(p-1)/(2n-2)+1} \equiv d_{n-1}A(0, \ldots, 0, (p-1)/(2n-2) + 1, 0).
\]

The coefficient is \( \equiv 0 (\mod p) \) so (2.2) follows. To check (2.3), let \( q = j \) and compare coefficients of \( x_j x_{n-1}^{(p-1)/(2n-2)-j} \).

If \( j < n \) the monomial can occur only in terms \( (-1)^{j+1} S_{(p-1)/2+j}(\xi) \) on the left of (2.1) and in \( d_j(-1)^{j+1} S_{(p-1)/2+j}(\gamma) \) on the right of (2.1). This gives (2.3) for \( 1 \leq j \leq n-1 \). As for the case of (2.3), \( j = n \), a brief digression will be necessary to show that either \( d_{n-1} \) or \( d_1, n-1 \equiv 0 (\mod p) \).

In (2.1) set \( q = n \) and compare the coefficients of \( x_1 x_{n-1}^{(p-1)/(2n-2)+1} \). By [4, Example 3, p. 81], the left side of (2.1) may be replaced with \( P_n(\xi) s_{(p-1)/2}(\xi) \); in this polynomial, mod \( H \), \( x_1 x_{n-1}^{(p-1)/(2n-2)+1} \) occurs only by taking the product of \( d_{n-1} x_1 x_{n-1} \) in \( P_n(\xi) \) with

\[
A(0, \ldots, 0, (p-1)/(2n-2), 0) d_{n-1}^{(p-1)/(2n-2)}(x_{n-1})^{(p-1)/(2n-2)} \text{ in } s_{(p-1)/2}(\xi).
\]
On the right we obtain $d_{1,n-1}$ with a factor
\[ A(0, \cdots, 0, (p-1)/(2n-2) + 1, 0) \]
\[ + A(1, 0, \cdots, 0, (p-1)/(2n-2), 0). \]

This shows:

\[ (2.4) \quad d_{1,n-1}(d_{n-1})^{(p-1)/(2n-2)}(n-1) \equiv d_{1,n-1}(n-1 + (p-1)/2 + 1). \]

From this $d_{1,n-1} \equiv 0$ (if $d_{n-1} \not\equiv 0$, then $d_{n-1}^{(p-1)/(2n-2)} \equiv 1$ by (2.2). Recall that $2n-2$ divides $p-1$ so that $n-1 \not\equiv 0 \pmod{p}$ and $n+(p-1)/2 \not\equiv 0 \pmod{p}$).

The remaining case of (2.3) may now be dealt with by letting $q = n$ in (2.1) and comparing coefficients of $x_n x_{n-1}^{(p-1)/(2n-2) - n}$. Since $d_{1,n-1}$ is known to be divisible by $p$, the polynomial $d_{1,n-1} x_{n-1} \mathcal{L}_{(p-1)/2}(x_1)$ in the right side of (2.1) (which occurs in $d_{1,n-1} \mathcal{L}_{(p-1)/2}(x_1 x_{n-1})$) may be disregarded. As in the preceding arguments only one term from each side of (2.1) has the desired monomial and the result follows.

By using an induction on the dimension of $\omega$ (and a subinduction on the number of components in $\omega$) the following may be shown:

(2.5) If $d_{n-1} \equiv 0$ then $d_{\omega} \equiv 0$ for all $\omega$.

By means of the same procedure it is possible to show:

(2.6) If $d_{n-1} \not\equiv 0$, then $d_{\omega} \equiv 0$ for all $\omega$

having more than one component.

Note that for both (2.5) and (2.6) the induction may be started with $d_{1,n-1} \equiv 0$, the consequence of (2.4).

(2.7) Lemma. If $d_{n-1} \not\equiv 0$ then $d_j \equiv d_j$.

Proof. Since $d_{n-1}^{(p-1)/(2n-2)} \equiv 1$, (2.2) becomes $d_j d_{n-1}^{(p-1)/(2n-2)} \equiv d_j^n$. Let $r$ be a primitive root mod $p$ and let $r^t_1 = d_j$. Thus

(2.8) $j t_n = t_j + j t_{n-1} \pmod{p-1}$.

Now in (2.8) set $j = 1$ and multiply by $j$, i.e. $j t_n \equiv j t_1 + j t_{n-1} \pmod{p-1}$ so that $j t_1 \equiv t_j \pmod{p-1}$.

3. Bundles over $\text{BSp}(n)$. In the preceding section it was assumed that $\dim X$ was finite. Now let $\xi \to X$ be the restriction of an $\text{Sp}(n)$ vector bundle over $\text{BSp}(n)$. The congruences in (2.5), (2.6) and (2.7) may be replaced by equality:

(3.1) Theorem. If $\xi_0$ is an $\text{Sp}(n)$ vector bundle over $\text{BSp}(n)$ then $p(\xi_0) = \sum_{j=1} d_j p_j(\gamma_0)$.

Proof. Suppose $p_\gamma(\xi_0) = \sum_{\omega \in \pi(\gamma)} d_\omega p_\omega(\gamma)$. Let $X$ be a skeleton of $\text{BSp}(n)$ of dimension $\geq 2(p-1+2n)$ where $p$ is some odd prime strictly
greater than \( \max\{n, d^q_\omega | \omega \in \pi(q), 1 \leq q \leq n\} \). If the inclusion is \( e:X \to B\text{Sp}(n) \), (2.5), (2.6) and (2.7) give, in \( H^*(X, \mathbb{Z}) \), \( e^*(p_\omega(\xi_0)) = p_\omega(\xi) = d^*_\omega p_\omega(\gamma) = d^*_\omega e^*(p_\omega(\gamma_0)) \). Since \( e^* \) is a monomorphism in these low dimensions the result follows.

Note that if \( \xi_0 \to B\text{Sp}(n) \) is an \( \text{Sp}(m) \) vector bundle and \( m<n \), then taking Whitney sum with a trivial real \( 4(n-m) \) bundle produces an \( \text{Sp}(n) \) vector bundle \( \theta \) having \( p_n(\theta) = 0 \). Then \( d_1 = 0 \) so \( \rho(\xi_0) = 1 \).

REFERENCES


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