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PONTRYAGIN CLASSES OF VECTOR BUNDLES OVER BSp(n)

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ABSTRACT. Let X be a finite skeleton of the classifying space of BSp(n), $\gamma_0 \to \mathrm{BSp}(n)$, the classifying bundle for Sp(n) vector bundles and $\gamma \to X$ the restriction of γ_0 over X. If $\xi \to X$ is another Sp(n) vector bundle, the Pontryagin classes $p_q(\xi)$ must be congruent to $d_1^q p_q(\gamma)$ modulo certain odd primes. Equality obtains if ξ is the restriction over X of a $\xi_0 \to \mathrm{BSp}(n)$. In particular, Sp($m$) vector bundles θ over BSp(n) have $p(\theta)=1$ if $m<n$.

1. **Notation.** Let X denote a skeleton of BSp(n), the classifying space for quaternionic n dimensional vector bundles over compact spaces. Let $P_q(\gamma)=x_q \in \Omega_{\mathrm{Sp}}^{4q}(X)$ denote the symplectic cobordism Pontryagin classes of the classifying bundle $\gamma_0 \to \mathrm{BSp}(n)$, pulled back to $\gamma \to X$ by the inclusion. Just as in [1, (8.1)], it can be shown that $1, x_1, x_2, \cdots, x_n$ generate $\Omega_{\mathrm{Sp}}^*(\mathrm{BSp}(n))$ as a free Ω_*^{Sp}-module. If $\omega = i_1, i_2, \cdots, i_t$ is a partition of q, let $x_\omega \in \Omega_{\mathrm{Sp}}^{4q}(X)$ denote the monomial $x_{i_1} x_{i_2} \cdots x_{i_t}$. Let $\pi(q)$ denote the set of all partitions of q.

Suppose ξ is an Sp(n) vector bundle over X. Then

$$(1.1) \quad P_q(\xi) = \sum_{\omega \in \pi(q)} d_\omega x_\omega + \text{terms having coefficients in } \Omega_i^{\mathrm{Sp}}, \qquad i > 0.$$

Since the Hurewicz transformation $\mu : \Omega_{\mathrm{Sp}}^*(\ ) \to H^*(\ ;Z)$ preserves characteristic classes, the summation term in (1.1) gives the usual Pontryagin classes of ξ, $p_q(\xi)$, in terms of the generators of $H^*(X;Z)$, $p_\omega(\gamma)$.

Recall (e.g. from [3, p. 179]) that Ω_*^{Sp} has $2p^i-2$ dimensional ring generators $[M^{p^i-1}]$ whose tangential characteristic numbers, $s_{(p^i-1)/2}(p(\tau(M)))$, are $\pm 2^h p$, p an odd prime, h some nonnegative integer. There is a relationship to the Landweber cohomology operations [2, §3 and §9],

$$\mathscr{L}_{(p^i-1)/2}([M^{p^i-1}]) = -s_{(p^i-1)/2}(p(\tau(M))).$$

In the following section, let p denote an odd prime such that $2n-2$ divides $p-1$. Let congruences between integers be modulo p unless otherwise indicated. The set of elements in $\Omega_{\mathrm{Sp}}^*(X)$ of the form αx_ω, where

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$\alpha \in \Omega_i^{\mathrm{Sp}}$ is such that either $i>0$ or $\alpha \in p\Omega_0^{\mathrm{Sp}}$, generates an ideal H. In what follows let congruences between elements of $\Omega_{\mathrm{Sp}}^*(X)$ be understood to be modulo H. For convenience the coefficients occurring in [**4**, Example 4, p. 82] will be denoted $A(\lambda_1, \lambda_2, \cdots, \lambda_n)$. If the sum of r powers $s_r = y_1^r + y_2^r + \cdots + y_n^r$ is expressed in terms of the elementary symmetric functions $\sigma_1, \sigma_2, \cdots, \sigma_n$ then

$$s_r(\sigma_1, \cdots, \sigma_n) = \sum A(\lambda_1, \cdots, \lambda_n)\sigma_1^{\lambda_1}(-\sigma_2)^{\lambda_2} \cdots (\pm\sigma_n)^{\lambda_n}.$$

$s_r(\xi)$ will denote s_r with σ_q replacing $P_q(\xi)$. Finally, suppose dimension X is greater than $2(p-1+2n)$.

2. **Restrictions** mod p. We apply the Landweber operation, $\mathscr{L}_{(p-1)/2}$, to (1.1) and use the independence of $\{x_\omega \mid \omega \in \pi(q), 1 \leqq q \leqq n\}$.

Consider $\mathscr{L}_{(p-1)/2}(P_q(\xi)) \equiv \sum d_\omega \mathscr{L}_{(p-1)/2}(x_\omega)$ or:

$$\begin{aligned}(2.1)\qquad &\sum_{t=1}^{q}(-1)^{t+1}P_{q-t}(\xi)s_{(p-1)/2+t}(\xi) \\ &\qquad \equiv \sum_{\omega\in\pi(q)} d_\omega \sum_{\alpha a=\omega} x_\alpha \sum_{t=1}^{a}(-1)^{t+1}x_{a-t}s_{(p-1)/2+t}(\gamma).\end{aligned}$$

$$(2.2)\qquad d_{n-1}^{(p-1)/(2n-2)} \equiv d_{n-1}.$$

$$(2.3)\qquad d_n^j(d_{n-1})^{(p-1)/(2n-2)-j} \equiv d_j \quad \text{if } d_{n-1} \not\equiv 0.$$

PROOF. In (2.1) compare the coefficients of $x_{n-1}^{(p-1)/(2n-2)+1}$ for $q=n-1$. On the left only $(-1)^n s_{(p-1)/2+n-1}(\xi)$ may have a monomial term $x_{n-1}^{(p-1)/(2n-2)+1}$; as for the right of (2.1), only $d_{n-1}(-1)^n s_{(p-1)/2+n-1}(\gamma)$ may. Thus

$$\begin{aligned}&A(0, \cdots, 0, (p-1)/(2n-2)+1, 0)(d_{n-1})^{(p-1)/(2n-2)+1} \\ &\qquad \equiv d_{n-1}A(0, \cdots, 0, (p-1)/(2n-2)+1, 0).\end{aligned}$$

The coefficient is $\not\equiv 0 \pmod p$ so (2.2) follows. To check (2.3), let $q=j$ and compare coefficients of $x_n^j x_{n-1}^{(p-1)/(2n-2)-j}$.

If $j<n$ the monomial can occur only in terms $(-1)^{j+1}s_{(p-1)/2+j}(\xi)$ on the left of (2.1) and in $d_j(-1)^{j+1}s_{(p-1)/2+j}(\gamma)$ on the right of (2.1). This gives (2.3) for $1 \leqq j \leqq n-1$. As for the case of (2.3), $j=n$, a brief digression will be necessary to show that either d_{n-1} or $d_{1,n-1} \equiv 0 \pmod p$.

In (2.1) set $q=n$ and compare the coefficients of $x_1 x_{n-1}^{(p-1)/(2n-2)+1}$. By [**4**, Example 3, p. 81], the left side of (2.1) may be replaced with $P_n(\xi)s_{(p-1)/2}(\xi)$; in this polynomial, mod H, $x_1 x_{n-1}^{(p-1)/(2n-2)+1}$ occurs only by taking the product of $d_{1,n-1}x_1 x_{n-1}$ in $P_n(\xi)$ with

$$A(0, \cdots, 0, (p-1)/(2n-2), 0)d_{n-1}^{(p-1)/(2n-2)}(x_{n-1})^{(p-1)/(2n-2)} \text{ in } s_{(p-1)/2}(\xi).$$

On the right we obtain $d_{1,n-1}$ with a factor

$$A(0,\cdots,0,(p-1)/(2n-2)+1,0)$$
$$+A(1,0,\cdots,0,(p-1)/(2n-2),0).$$

This shows:

$$(2.4)\quad d_{1,n-1}(d_{n-1})^{(p-1)/(2n-2)}(n-1)\equiv d_{1,n-1}(n-1+(p-1)/2+1).$$

From this $d_{1,n-1}\equiv 0$ (if $d_{n-1}\not\equiv 0$, then $d_{n-1}^{(p-1)/(2n-2)}\equiv 1$ by (2.2). Recall that $2n-2$ divides $p-1$ so that $n-1\not\equiv 0 \pmod p$ and $n+(p-1)/2\not\equiv 0 \pmod p$)).

The remaining case of (2.3) may now be dealt with by letting $q=n$ in (2.1) and comparing coefficients of $x_n x_{n-1}^{(p-1)/(2n-2)-n}$. Since $d_{1,n-1}$ is known to be divisible by p, the polynomial $d_{1,n-1}x_{n-1}\mathscr{L}_{(p-1)/2}(x_1)$ in the right side of (2.1) (which occurs in $d_{1,n-1}\mathscr{L}_{(p-1)/2}(x_1x_{n-1})$) may be disregarded. As in the preceding arguments only one term from each side of (2.1) has the desired monomial and the result follows.

By using an induction on the dimension of ω (and a subinduction on the number of components in ω) the following may be shown:

$$(2.5)\qquad \text{If } d_{n-1}\equiv 0 \text{ then } d_\omega\equiv 0 \text{ for all } \omega.$$

By means of the same procedure it is possible to show:

$$(2.6)\qquad \begin{array}{l}\text{If } d_{n-1}\not\equiv 0, \text{ then } d_\omega\equiv 0 \text{ for all } \omega\\ \text{having more than one component.}\end{array}$$

Note that for both (2.5) and (2.6) the induction may be started with $d_{1,n-1}\equiv 0$, the consequence of (2.4).

(2.7) LEMMA. *If $d_{n-1}\not\equiv 0$ then $d_1^j\equiv d_j$.*

PROOF. Since $d_{n-1}^{(p-1)/(2n-2)}\equiv 1$, (2.2) becomes $d_jd_{n-1}^j\equiv d_n^j$. Let r be a primitive root mod p and let $r^{t_i}=d_i$. Thus

$$(2.8)\qquad jt_n=t_j+jt_{n-1}\pmod{p-1}.$$

Now in (2.8) set $j=1$ and multiply by j, i.e. $jt_n\equiv jt_1+jt_{n-1} \pmod{p-1}$ so that $jt_1\equiv t_j \pmod{p-1}$.

3. **Bundles over** BSp(n). In the preceding section it was assumed that dim X was finite. Now let $\xi\to X$ be the restriction of an Sp(n) vector bundle over BSp(n). The congruences in (2.5), (2.6) and (2.7) may be replaced by equality:

(3.1) THEOREM. *If ξ_0 is an* Sp(n) *vector bundle over* BSp(n) *then* $p(\xi_0)=\sum_{j=1}^n d_1^jp_j(\gamma_0)$.

PROOF. Suppose $p_q(\xi_0)=\sum_{\omega\in\pi(q)}d_\omega p_\omega(\gamma)$. Let X be a skeleton of BSp(n) of dimension $\geqq 2(p-1+2n)$ where p is some odd prime strictly greater than $\max\{n, d_\omega^q | \omega \in \pi(q), 1 \leqq q \leqq n\}$. If the inclusion is $e: X \to \mathrm{BSp}(n)$, (2.5), (2.6) and (2.7) give, in $H^*(X, \mathbf{Z})$, $e^*(p_q(\xi_0)) = p_q(\xi) = d_1^q p_q(\gamma) = d_1^q e^*(p_q(\gamma_0))$. Since e^* is a monomorphism in these low dimensions the result follows.

Note that if $\xi_0 \to \mathrm{BSp}(n)$ is an $\mathrm{Sp}(m)$ vector bundle and $m < n$, then taking Whitney sum with a trivial real $4(n-m)$ bundle produces an $\mathrm{Sp}(n)$ vector bundle θ having $p_n(\theta) = 0$. Then $d_1 = 0$ so $p(\xi_0) = 1$.

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