Beurling Generalized Prime Number Systems in Which the Chebyshev Inequalities Fail

R. S. Hall

Abstract. It is proved that there exist systems of generalized primes in which the asymptotic distribution of integers is \( N(x) = Ax + O(x \cdot \log^{-\gamma} x) \) with \( A > 0 \) and \( \gamma \in [0, 1) \) and in which the Chebyshev inequalities

\[
\liminf_{x \to \infty} \frac{\pi(x) \log x}{x} > 0, \quad \limsup_{x \to \infty} \frac{\pi(x) \log x}{x} < \infty
\]

do not hold.

A nondecreasing sequence \( P \) of real numbers \( p_1, p_2, \cdots \) such that \( p_1 > 1 \) and \( p_i \to \infty \) is called a Beurling generalized prime number system. The associated system of generalized integers, \( N = \{n_i\}_{i=0}^\infty \), is the sequence of real numbers obtained by letting \( n_0 = 1 \) and arranging in nondecreasing order the elements of the multiplicative semigroup generated by \( P \). The distribution functions \( \pi(x) \) and \( N(x) \) are defined in the natural way,

\[
\pi(x) = \pi_P(x) = \sum_{p_i \leq x} 1, \quad N(x) = N_P(x) = \sum_{n_i \leq x} 1.
\]

Beurling [2] proved that if

\[
N(x) = Ax + O(x \log^{-\gamma} x)
\]

with \( A > 0 \) and \( \gamma > \frac{3}{2} \), then the prime number theorem holds for the system, that is, \( \pi(x) \sim x / \log x \). He also showed, essentially, that there exist systems for which (1) holds with \( \gamma = \frac{3}{2} \) and in which the theorem is not true. A summary of results and conjectures about systems of generalized primes is given in the recent work of Bateman and Diamond [1].

It is easy to show (as was done by the author in [3]) that if \( N(x) = Ax + o(x) \), then

\[
\liminf_{x \to \infty} \frac{\pi(x) \log x}{x} \leq 1, \quad \limsup_{x \to \infty} \frac{\pi(x) \log x}{x} \geq 1.
\]
In the following article by Diamond the Chebyshev inequalities
\[
\liminf_{x \to \infty} \frac{\pi(x) \log x}{x} \geq a > 0, \quad \limsup_{x \to \infty} \frac{\pi(x) \log x}{x} \leq A < \infty
\]
are shown to hold when (1) is satisfied with \( \gamma > 1 \). In this paper we show that these Chebyshev inequalities need not hold when \( \gamma < 1 \).

**Theorem.** Let \( \alpha \in [0, 1] \), \( \beta \in [1, +\infty) \), and \( \gamma \in [0, 1) \) be given. There exists a generalized prime number system in which

(i) \( N(x) = Ax + O(x \log^{-\gamma} x) \),

(ii) \( \liminf_{x \to +\infty} \frac{\pi(x) \log x}{x} = \alpha \),

(iii) \( \limsup_{x \to +\infty} \frac{\pi(x) \log x}{x} = \beta \).

**Proof.** Let \( \pi_0(x) \) be the distribution function for the rational primes. For each rational integer \( n \) define the interval \( I_n = (a_n, b_n] \) by \( \log^{1-\gamma} a_n = 2^n \) and \( b_n = a_n 2^{n/4} \). Let \( C_n \) be the set consisting of the largest \([ (1-\alpha) (\pi_0(b_n) - \pi_0(a_n)) ] \) rational primes in \( I_n \). If \( \beta < +\infty \), let \( D_n \) be a set consisting of any \([ (\beta - 1) (\pi_0(b_n) - \pi_0(a_n)) ] \) distinct real numbers in the interval \( (b_n-1, b_n] \). If \( \beta = +\infty \), let \( D_n \) be a set consisting of any \([ 2^{n/8} (\pi_0(b_n) - \pi_0(a_n)) ] \) distinct real numbers in \( (b_n-1, b_n] \). Let \( C = \bigcup_{n \in \text{odd}} C_n \), \( D = \bigcup_{n \in \text{even}} D_n \), and let \( P = \{ p_{i} \}_{i=1}^{\infty} \) be the nondecreasing sequence formed from the set \( P = (R - C) \cup D \), where \( R \) is the set of rational primes.

Then for some \( K = K(\beta) \) we have
\[
\sum_{p \in C \cup D} \frac{\log^\gamma p}{p} \leq \sum_{n \in \text{odd}} \left\{ \pi_0(b_n) - \pi_0(a_n) \right\} \frac{\log^\gamma a_n}{a_n} + K \sum_{n \in \text{even}} \frac{2^{n/8} (\pi_0(b_n) - \pi_0(a_n)) \log^\gamma a_n}{a_n} = O\left( \sum_{n=1}^{\infty} \frac{1}{2^{n/2}} \right) = O(1).
\]

Thus we see that \( \prod_{p \in C} \left( 1 + \frac{\log^\gamma p}{p} \right) \prod_{q \in D} \left( 1 - \frac{\log^\gamma q}{q} \right)^{-1} \) converges. If for \( \Re s > 1 \) we define \( \{ c_k \}_{k=1}^{+\infty} \) by
\[
\sum_{k=1}^{+\infty} c_k^{s} = \prod_{p \in C} \left( 1 - \frac{1}{p^s} \right) \prod_{q \in D} \left( 1 - \frac{1}{q^s} \right)^{-1},
\]
then we have
\[
(2) \quad \sum_{k=1}^{+\infty} \frac{|c_k| \log^\gamma k}{k} \leq \prod_{p \in C} \left( 1 + \frac{\log^\gamma p}{p} \right) \prod_{q \in D} \left( 1 - \frac{\log^\gamma q}{q} \right)^{-1}
\]
by repeated use of the inequality $\log^\gamma(mn) = (\log m + \log n)^\gamma \leq \log^\gamma m + \log^\gamma n$.

Define for $\text{Re } s > 1$ the function

$$
\zeta(s) = \zeta_0(s) \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^s}\right) \prod_{q \in \mathcal{D}} \left(1 - \frac{1}{q^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{a_n}{n^s},
$$

where $\zeta_0(s)$ is the Riemann zeta function. Then $\sum_{n=1}^{\infty} a_n/n^s = (\sum_{m=1}^{\infty} 1/m^s) \sum_{k=1}^{\infty} c_k/k^s$ and

$$
N(x) = \sum_{n \leq x} a_n = \sum_{k \leq x} c_k \left[\frac{x}{k}\right] = x \sum_{k \leq x} \frac{c_k}{k} + O \left(\sum_{k \leq x} \left|c_k\right|\right)
$$

$$
= x \sum_{k=1}^{\infty} \frac{c_k}{k} + O \left(x \sum_{k > x} \frac{|c_k|}{k}\right) + O \left(\sum_{k \leq x} \left|c_k\right|\right).
$$

From (2) we have

$$
\log^\gamma x \sum_{k > x} \frac{|c_k|}{k} \leq \sum_{k > x} \frac{|c_k| \log^\gamma k}{k} = O(1)
$$

and

$$
\sum_{k \leq x} |c_k| \leq O(1) + \frac{x}{\log^\gamma x} \sum_{k \leq x} \frac{|c_k| \log^\gamma k}{k} = O\left(\frac{x}{\log^\gamma x}\right).
$$

Thus $N(x) = x \sum_{k=1}^{\infty} c_k/k + O(x/\log^\gamma x)$.

We next show that (ii) and (iii) hold. Since $\pi_0(x) \sim x/\log x$ we have

$$
\lim_{n \to \infty} \frac{\pi_0(a_n)}{\pi_0(b_n)} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{\pi_0(b_{n-1})}{\pi_0(a_n)} 2^{n/8} = 0.
$$

From the fact that

$$
\pi(x) \geq \pi_0(x) - \sum_{b_n \leq x; n \text{ odd}} [(1 - \alpha)\{\pi_0(b_n) - \pi_0(a_n)\}]
$$

$$
\geq \pi_0(x) - (1 - \alpha)\pi_0(x)
$$

it is clear that

$$
\lim \inf_{x \to +\infty} \frac{\pi(x) \log x}{x} = \lim \inf_{x \to +\infty} \frac{\pi(x)}{\pi_0(x)} \geq \alpha.
$$

On the other hand

$$
\pi(a_n) \leq \pi_0(a_n) + K \sum_{k < n; k \text{ even}} 2^{k/8}\{\pi_0(b_k) - \pi_0(a_k)\}
$$

$$
\leq \pi_0(a_n) + K 2^{n/8} \pi_0(b_{n-1}).
$$
It follows then that \( \lim_{n \to \infty} \frac{\pi(a_n)}{\pi_0(b_n)} = 0 \) and we have (i), since

\[
\liminf_{x \to \infty} \frac{\pi(x)}{\pi_0(x)} \leq \liminf_{n \to \infty; n \text{ odd}} \frac{\pi(b_n)}{\pi_0(b_n)} = \liminf_{n \to \infty; n \text{ odd}} \frac{\pi(a_n) + \alpha \{\pi_0(b_n) - \pi_0(a_n)\}}{\pi_0(b_n)} = \alpha.
\]

If \( \beta < \infty \), then

\[
\pi(x) \leq \pi_0(x) + \sum_{b_n \leq x; n \text{ even}} (\beta - 1) \{\pi_0(b_n) - \pi_0(a_n)\} \leq \beta \pi_0(x).
\]

Similarly, it follows that

\[
\limsup_{x \to \infty} \frac{\pi(x)}{\pi_0(x)} \geq \limsup_{n \to \infty; n \text{ even}} \frac{\pi(b_n)}{\pi_0(b_n)} = \limsup_{n \to \infty; n \text{ even}} \frac{\pi(a_n) + \beta \{\pi_0(b_n) - \pi_0(a_n)\}}{\pi_0(b_n)} = \beta.
\]

Thus, \( \limsup_{x \to \infty} \pi(x) \log x/x = \beta \). If \( \beta = \infty \), then

\[
\limsup_{x \to \infty} \frac{\pi(x)}{\pi_0(x)} \geq \limsup_{n \to \infty} \frac{\pi(b_n)}{\pi_0(b_n)} = \limsup_{n \to \infty} \frac{\pi(a_n) + (2^{n/8} + 1) \{\pi_0(b_n) - \pi_0(a_n)\}}{\pi_0(b_n)} = \infty.
\]

This completes the proof.

A slightly more complicated argument shows that (ii) and (iii) can hold for systems in which \( N(x) = Ax + O(x/g(x)) \) provided \( g(x) = o(\log x) \). Details may be found in [3].

REFERENCES


**Department of Mathematics, Willamette University, Salem, Oregon 97301**