PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 1, September 1973

EQUATIONS WHICH CHARACTERIZE INNER PRODUCT SPACES

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ABSTRACT. It is shown that if N is a normed linear space and there is a point y of norm 1 such that an inequality of the type $a^2\|x\|^2 \leqq \lim_{u\to 0} G(\{\|b_i ux + c_i y\|\}_{i=1}^n) \leqq b^2\|x\|^2$ holds for all x in N (where $0 < a \leqq b$, the c_i's are nonzero and G and $\|\cdot\|$ satisfy a certain twice-differentiability condition), then N is isomorphic to an inner product space and $\inf \|T\| \cdot \|T^{-1}\| \leqq b/a$, where the infimum is taken over all linear homeomorphisms T between N and an inner product space. In the event that $a=b=1$, the inequality reduces to an equation which characterizes inner product spaces. An example shows that these results do not follow without the twice-differentiability condition on G.

S. O. Carlsson has proved [1] that a normed linear space N is an inner product space if an equation of the type

$$\sum_{i=0}^{n} a_i \|b_i x + c_i y\|^2 = 0 \tag{1}$$

(where the numbers a_i are nonzero and the couples (b_i, c_i) are pairwise linearly independent) holds for all x and y in N. An example of this type of equation is the Jordan-von Neumann condition for inner product spaces [2]:

$$\|x+y\|^2 + \|x-y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2 \tag{2}$$

for all x and y in N. A reformulation of Carlsson's condition is the following:

$$\|x\|^2 = \sum_{i=1}^{n} a_i \|b_i x + c_i y\|^2 \tag{3}$$

(where the coefficients are different from those in (1) and the numbers c_i are nonzero) for all x and y in N. Now, if we define a function G_0 by $G_0(t_1, t_2, \cdots, t_n) = \sum_{i=1}^n a_i t_i^2$, then equation (3) says that, for all x and y in N,

$$\|x\|^2 = G_0(\{\|b_i x + c_i y\|\}_{i=1}^n). \tag{4}$$

Received by the editors October 16, 1972.
AMS (MOS) subject classifications (1970). Primary 46C10; Secondary 39A25.

In this paper Carlsson's condition is improved upon in several ways simultaneously: the specific function G_0 is replaced by a function G which only need be twice-differentiable at a certain point, the requirement that (4) hold for all y is replaced by the condition that it hold for a single y (on the unit sphere for convenience) where the norm is twice-differentiable, the requirement that (4) hold for all x is replaced by a weaker limit condition, and the equality in (4) is replaced by inequalities which give information on the nearness of N to an inner product space in case N is isomorphic but not isometric to an inner product space. The main result of this paper is: If there is a point y on the unit sphere and some function G such that an inequality of the type

$$a^2 \|x\|^2 \leqq \lim_{u\to 0} \frac{G(\{\|b_i ux + c_i y\|\}_{i=1}^n)}{u^2} \leqq b^2 \|x\|^2 \tag{5}$$

holds for all x in N (where $0<a\leqq b$, the numbers c_i are nonzero, and G and the norm satisfy a certain twice-differentiability condition), then N is isomorphic to an inner product space. Furthermore, letting $K(N)=\inf\{\|T\|\cdot\|T^{-1}\| : T \text{ is an isomorphism from } N \text{ onto an inner product space}\}$, it follows that $K(N)\leqq b/a$. (This means that an inner product $((\cdot,\cdot))$ exists such that $a^2\|x\|^2\leqq((x,x))\leqq b^2\|x\|^2$ for all x in N.) An example shows that these results do not follow without the twice-differentiability condition on G. The twice-differentiability condition on the norm can be removed if we require that (5) hold for all x and y in N.

Say that a function G from a subset A of a normed linear space N_1 into a normed linear space N_2 is twice-differentiable at the point x of A if x is a limit point of A and their exist a bounded linear mapping T_1 from N_1 into N_2 and a bounded bilinear mapping T_2 from $N_1\times N_1$ into N_2 such that

$$\lim_{\varepsilon\to 0,\, x+\varepsilon\in A} \frac{G(x+\varepsilon)-G(x)-T_1(\varepsilon)-\frac{1}{2}T_2(\varepsilon,\varepsilon)}{\|\varepsilon\|^2}=0.$$

This definition does not require that G be differentiable at points of A other than x. The necessary lemmas on this subject are stated at the end.

Throughout this paper, N denotes a normed linear space. An n-tuple $(t_1, t_2, \cdots, t_n)$ is denoted $\{t_i\}$. If $t_0, t_1, \cdots, t_n$ are numbers or points, $\{t_i\}$ denotes $(t_1, t_2, \cdots, t_n)$, not $(t_0, t_1, \cdots, t_n)$.

THEOREM 1. *Suppose that there exist a function G from N into $\mathbf{R}$ which is twice-differentiable at* 0 *and numbers a and b, $0<a\leqq b$, such that for all x in N, $a^2\|x\|^2\leqq\lim_{u\to 0} G(ux)/u^2\leqq b^2\|x\|^2$. Then $K(N)\leqq b/a$.*

PROOF. Let T_1 and T_2 be derivatives of G at 0, as in the above definition.

Then

$$\lim_{x\to 0} \frac{G(x) - G(0) - T_1(x) - \frac{1}{2}T_2(x, x)}{\|x\|^2} = 0.$$

This implies that G is continuous at 0, and since $\lim_{u\to 0} G(ux)/u^2$ exists, $G(0)=0$. For all x in N, $0=\lim_{u\to 0}(G(ux)-uT_1(x)-\frac{1}{2}u^2T_2(x,x))/u^2=(\lim_{u\to 0} G(ux)/u^2)-\frac{1}{2}T_2(x,x)-(\lim_{u\to 0} T_1(x)/u)$. Therefore, $T_1=0$ and $\lim_{u\to 0} G(ux)/u^2=\frac{1}{2}T_2(x,x)$. For all x and y in N, define $((x,y))=(T_2(x,y)+T_2(y,x))/4$. Then $((\cdot,\cdot))$ is an inner product for N satisfying $a^2\|x\|^2\leqq((x,x))\leqq b^2\|x\|^2$ for all x in N. This implies $K(N)\leqq b/a$.

COROLLARY 1.1. *If the square of the norm is twice-differentiable at* 0, *then N is an inner product space.*

THEOREM 2. *Suppose that there exist a point y of norm* 1 *where $\|\cdot\|$ is twice-differentiable, numbers b_i and c_i, $c_i\neq 0$, $i=1,\cdots,n$, a function G from a subset of $\mathbf{R}^n$ into $\mathbf{R}$ twice-differentiable at $\{|c_i|\}$, and numbers a and b, $0<a\leqq b$, such that for every x in N, $a^2\|x\|^2\leqq\lim_{u\to 0} G(\{\|b_iux+c_iy\|\})/u^2\leqq b^2\|x\|^2$. Then $K(N)\leqq b/a$.*

PROOF. Let $\mathcal{N}=N_1\times N_2\times\cdots\times N_n$, where each $N_i=N$. Then define functions $E:\mathbf{R}\to\mathcal{N}$ and $F:\mathcal{N}\to\mathbf{R}^n$ by $E(u)=\{b_iux+c_iy\}$ and $F(\{p_i\})=\{\|p_i\|\}$. Then E is twice-differentiable at 0, F is twice-differentiable at $E(0)$, and G is twice-differentiable at $F\circ E(0)$. By two applications of Lemma 1 on twice-differentiable functions, $G\circ F\circ E$ is twice-differentiable at 0. By the hypothesis $a^2\|x\|^2\leqq\lim_{u\to 0} G\circ F\circ E(ux)/u^2\leqq b^2\|x\|^2$ for all x in N. Theorem 1 then asserts that $K(N)\leqq b/a$.

One immediate consequence of Theorem 2 is the following improvement on the Jordan-von Neumann condition:

COROLLARY 2.1. *If there exist a point y of norm* 1 *in N where $\|\cdot\|$ is twice-differentiable and numbers a and b, $0<a\leqq b$, such that for all x in N,*

$$a^2\|x\|^2 \leqq \lim_{u\to 0}\frac{\|ux+y\|^2+\|ux-y\|^2-2\|y\|^2}{2u^2}\leqq b^2\|x\|^2,$$

then $K(N)\leqq b/a$.

PROOF. It is just a matter of checking that the function $G(t_1,t_2,t_3)=(t_1^2+t_2^2-2t_3^2)/2$ is twice-differentiable at $(1, 1, 1)$.

If N is 2-dimensional, then Lemma 2 on twice-differentiable functions may be used to eliminate from the hypothesis of Corollary 2.1 the condition that there exist a point y of norm 1 where $\|\cdot\|$ is twice-differentiable.

COROLLARY 2.2. *Suppose that there exist a point y of norm* 1 *where $\|\cdot\|$ is twice-differentiable, pairwise linearly independent points in the plane*

(b_i, c_i), $i=0, 1, \cdots, n$, *a function* G *twice-differentiable at*

$$\{|b_0c_i - c_0b_i|/(b_0^2 + c_0^2)^{1/2}\},$$

and numbers a *and* b, $0<a\leqq b$, *such that for all* p *and* q *in* N, $a^2\|b_0p+c_0q\|^2\leqq G(\{\|b_ip+c_iq\|\})\leqq b^2\|b_0p+c_0q\|^2$. *Then* $K(N)\leqq b/a$.

PROOF. Define $b_i'=(b_0b_i+c_0c_i)/(b_0^2+c_0^2)^{1/2}$, $c_i'=(b_0c_i-c_0b_i)/(b_0^2+c_0^2)^{1/2}$, $i=0, 1, \cdots, n$. Then the c_i' are nonzero and $(b_0', c_0')=(1, 0)$. Suppose x is in N. Let $p=(b_0x-c_0y)/(b_0^2+c_0^2)^{1/2}$ and $q=(c_0x+b_0y)/(b_0^2+c_0^2)^{1/2}$. Then

$$\begin{aligned} a^2\|x\|^2 &= a^2\|b_0p + c_0q\|^2 \leqq G(\{\|b_ip + c_iq\|\}) \\ &= G(\{\|b_i'x + c_i'y\|\}) \leqq b^2\|b_0p + c_0q\|^2 = b^2\|x\|^2. \end{aligned}$$

By Theorem 2, $K(N)\leqq b/a$.

Lemma 2 on twice-differentiable functions may be used to omit from the hypothesis of the above theorem the condition on the existence of the point y if the conclusion is altered to read "$K(N')\leqq b/a$ for every two-dimensional subspace N' of N." However, this does not insure that $K(N)\leqq b/a$.

To get Carlsson's theorem from Corollary 2.2, define

$$G(t_1, t_2, \cdots, t_n) = \frac{-1}{a_0}\sum_{i=1}^{n} a_it_i^2$$

and note that G is twice-differentiable everywhere. Then, using Lemma 2 on twice-differentiable functions, we have that every 2-dimensional subspace of N is an inner product space, and, hence, that N is an inner product space.

If $n=3$, then the condition that G be twice-differentiable may be omitted from the hypothesis of Theorem 2, as this author has shown in [**3**, Theorem 6]. One is tempted to guess that for other values of n that condition is also unnecessary. The following example disproves that conjecture.

In the plane let A be the arc $\{(a, b): b\geqq 0, a^2+(b+\frac{1}{2})^2=1\}$ of the circle with radius 1 and center $(0, -\frac{1}{2})$. Let S be the unit sphere $A\cup(-A)$ and let $\|\cdot\|$ be the corresponding norm (the Minkowski functional for S). If S' is any linear image of S distinct from S, then S and S' have no more than 3 pairwise linearly independent points in common. Let (a_i, b_i), $i=0, 1, 2, 3, 4$, be five pairwise linearly independent points. We show that there is a function G such that, for every two points x and y,

$$G(\{\|a_ix + b_iy\|\}) = \|a_0x + b_0y\|^2.$$

Let x and y be points in the plane. By the strict convexity of S, x and y are linearly dependent if and only if there exist numbers c and d such that

$\|a_ix+b_iy\|=|a_ic+b_id|$, $i=1, 2, 3, 4$, and in this case, define

$$G(\{\|a_ix + b_iy\|\}) = (a_0c + b_0d)^2,$$

which is $\|a_0x+b_0y\|^2$. Suppose x and y are linearly independent. The four points $(a_i, b_i)/\|a_ix+b_iy\|$, $i=1, 2, 3, 4$, belong to $T(S)$, where T is the linear mapping such that $T(x)=(1, 0)$ and $T(y)=(0, 1)$. There exists no other linear image of S containing these points. Let k denote the positive number such that $(a_0, b_0)/k \in T(S)$. Then

$$1 = \|T^{-1}((a_0, b_0)/k)\| = \|a_0x + b_0y\|/k,$$

so $k=\|a_0x+b_0y\|$. Thus for x and y linearly independent, the rule is $G(\{\|a_ix+b_iy\|\})=k^2$, where k is the positive number such that $(a_0, b_0)/k$ belongs to the unique linear image of S which contains $(a_i, b_i)/\|a_ix+b_iy\|$, $i=1, 2, 3, 4$. Although such a function G exists, N is not an inner product space.

We have used the following two lemmas on twice-differentiable functions. The proof of Lemma 1 is omitted since it is quite similar to the proof of the ordinary composition theorem ("chain rule") in differential calculus.

LEMMA 1. *Suppose that F is a function from a subset A of the normed linear space N_1 into the normed linear space N_2, x is a point of A where F is twice-differentiable, G is a function from a subset of N_2 containing $F(x)$ into a normed linear space N_3, G is twice-differentiable at $F(x)$, and x is a member and limit point of* dom $F \circ G$. *Let S_1 and S_2 (S_1 linear and S_2 bilinear) be functions satisfying the definition of twice-differentiability for F at x, and let T_1 and T_2 be functions satisfying the definition for G at $F(x)$. Then $G \circ F$ is twice-differentiable at x and has derivatives $U_1=T_1 \circ S_1$ and $U_2=T_1 \circ S_2+T_2 \circ (S_1, S_1)$.*

LEMMA 2. *Every norm defined on the plane is twice-differentiable almost everywhere.*

PROOF. Suppose $\|\cdot\|$ is a norm defined on the plane. Let x and y be two linearly independent points of the unit sphere, and let r be the positive function such that $\|r(\theta) \cos(\theta)x+r(\theta) \sin(\theta)y\|=1$ for all θ. Then r is left-differentiable and for every θ, $-\arctan(r'_-(\theta)/r(\theta))+\theta+\pi/2$ gives the direction of the left-hand tangent to the unit sphere at $r(\theta) \cos(\theta)x+r(\theta) \sin(\theta)y$, so the function $-\arctan(r'_-(\theta)/r(\theta))+\theta+\pi/2$ is nondecreasing and differentiable almost everywhere. It turns out that r is twice-differentiable wherever r'_- is differentiable.

Suppose r'_- is differentiable at θ. Then r'_- is continuous at θ and r is differentiable at θ. Let T_1 be the linear function and T_2 the bilinear function defined by $T_1(\varepsilon)=\varepsilon r'(\theta)$ and $T_2(\varepsilon, \varepsilon)=\varepsilon^2(r'_-)'(\theta)$. For every ε, let

$h(\varepsilon)=r(\theta+\varepsilon)-r(\theta)-T_1(\varepsilon)$. Then $h(0)=h'(0)=0$, h is left-differentiable, and h'_- is differentiable at 0. It is a theorem that, under these conditions, $\lim_{\varepsilon\to 0}h(\varepsilon)/\varepsilon^2=(h'_-)'(0)/2$. This implies that

$$\lim_{\varepsilon\to 0}\frac{r(\theta+\varepsilon)-r(\theta)-T_1(\varepsilon)-\frac{1}{2}T_2(\varepsilon,\varepsilon)}{|\varepsilon|^2}=0,$$

so r is twice-differentiable almost everywhere.

The function f from the plane into $\boldsymbol{R}$ defined by

$$f(k\cos(\theta)x+k\sin(\theta)y)=\theta$$

if $k>0$ and $\theta\in[0,2\pi)$ is twice-differentiable except on the ray $\{kx:k\geqq 0\}$, and, hence, $r\circ f$ and $(r\circ f)^{-1}$ and the function $\|p\|=|p|/(r\circ f(p))$ are twice-differentiable almost everywhere.

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