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REAL-LINEAR OPERATORS ON QUATERNIONIC HILBERT SPACE

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ABSTRACT. The main result is that any continuous real-linear operator A on a quaternionic Hilbert space has a unique decomposition $A=A_0+i_1A_1+i_2A_2+i_3A_3$, where the A_ν are continuous linear operators and (i_1, i_2, i_3) is any right-handed orthonormal triad of vector quaternions. Other results concern the place of the colinear and complex-linear operators in this characterisation and the effect on the A_ν of a rotation of the triad of vector quaternions. A new result concerning symplectic images of a quaternionic Hilbert space is also presented.

1. **Introduction.** The study of group representations on quaternionic Hilbert space, reviewed in [**1**], makes extensive use of the complex Hilbert space structure that can be imposed on a quaternionic Hilbert space. This is also true of the recent work by Viswanath [**3**] on normal linear operators. In the present work the real Hilbert space structure is also considered, with particular attention to real-linear operators. The main result is that the class of continuous real-linear operators on a quaternionic Hilbert space $\mathscr{H}$ is simply the quaternion algebra generated by the real algebra of continuous linear operators on $\mathscr{H}$.

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2. **Preliminaries.** The division ring $\mathscr{Q}$ of quaternions is the real algebra generated by the elements of the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$. Thus any quaternion q can be written in the form:

$$q = q_0 + q_1 i + q_2 j + q_3 k \qquad (q_\nu \in \boldsymbol{R};\ \nu = 0, 1, 2, 3).$$

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The elements of the real subspace of $\mathcal{Q}$ generated by 1 are called *real quaternions*, while the elements of the subspace generated by $\{i, j, k\}$ are called *vector quaternions*. Since the latter subspace can be identified with $\boldsymbol{R}^3$ we often write $q=q_0+\xi$ ($\xi \in \boldsymbol{R}^3$) or $q=\sum_{\nu=0}^{3} q_\nu i_\nu$, where $i_0=1$ and (i_1, i_2, i_3) is any right-handed orthonormal triad in $\boldsymbol{R}^3$. Such a triad is called a *quaternion basis.*

The relationship between two quaternion bases (i_1, i_2, i_3) and (i_1', i_2', i_3') can be expressed by means of a 3×3 real orthogonal matrix $L=[l_{\mu\nu}]$, viz: $[i_1'\, i_2'\, i_3']=[i_1\, i_2\, i_3]L$, i.e. $i_\mu'=\sum_{\lambda=1}^{3} i_\lambda l_{\lambda\mu}$ ($\mu=1, 2, 3$).

Thus, if $q=\sum_{\nu=0}^{3} q_\nu i_\nu=\sum_{\nu=0}^{3} q_\nu' i_\nu'$, then $[q_0', q_1', q_2', q_3']=[q_0, q_1, q_2, q_3]M$, where M is the 4×4 matrix

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & L \end{array}\right].$$

Alternatively, the rotation in $\boldsymbol{R}^3$ defined by L can be represented by an inner automorphism ω of $\mathcal{Q}$, viz: $i_\mu'=qi_\mu q^{-1}$ ($\mu=1, 2, 3$), where q is a unit quaternion. This follows from the well-known fact that the continuous automorphisms of the division ring $\mathcal{Q}$ are all inner automorphisms and are in one-to-one correspondence with the rotations in $\boldsymbol{R}^3$.

The *canonical anti-automorphism* σ *of* $\mathcal{Q}$, called *quaternion conjugation*, is defined by

$$q^\sigma = (q_0 + \xi)^\sigma = q_0 - \xi, \qquad \forall q \in \mathcal{Q}.$$

3. **Real and complex Hilbert space structures.** Let $\mathscr{H}$ be a quaternionic Hilbert space with inner product $\langle\ ,\ \rangle$. It will sometimes be convenient to decompose this inner product as follows:

$$\langle x, y\rangle = \sum_{\nu=0}^{3} i_\nu \langle x, y\rangle_\nu, \qquad \forall x, y \in \mathscr{H},$$

where $\langle x, y\rangle_\nu \in \boldsymbol{R}$ ($\nu=0, 1, 2, 3$). It is easy to verify that $\langle x, y\rangle_\nu=\langle x, i_\nu y\rangle_0$ ($\nu=1, 2, 3$).

It is well known (see, for example [3]) that $\mathscr{H}$ can be given the structure of a complex Hilbert space. This is done by identifying the complex field $\boldsymbol{C}$ with the subfield $\boldsymbol{C}(i_1)$ of $\mathcal{Q}$, where $\boldsymbol{C}(i_1)=\{q_0+q_1 i_1 : q_0, q_1 \in \boldsymbol{R}\}$, and defining the complex inner product $(\ ,\ )$ by $(x, y)=\langle x, y\rangle_0+i_1\langle x, y\rangle_1$, $\forall x, y \in \mathscr{H}$. The resulting complex Hilbert space $\mathscr{H}_1(i_1)$ is called the *symplectic image* of $\mathscr{H}$. It can be verified that, if S is an orthonormal basis for $\mathscr{H}$, then the set $S_1=\{y, i_2 y : y \in S\}$ is an orthonormal basis for $\mathscr{H}_1(i_1)$.

The notation $\mathscr{H}_1(i_1)$ has been chosen to emphasise the fact that the complex inner product depends on the choice of unit vector quaternion i_1. However, there is a simple relationship between any two complex inner products defined in this way.

THEOREM 1. *Let u and u' be two unit vector quaternions and let* $(\ ,\ )$ *and* $(\ ,\ )'$ *be the natural images in* $\mathbf{C}$ *of the inner products on* $\mathscr{H}_1(u)$ *and* $\mathscr{H}_1(u')$ *respectively. If q is a unit quaternion such that $u'=quq^\sigma$, then*

$$(x, y)' = (q^\sigma x, q^\sigma y), \qquad \forall x, y \in \mathscr{H}. \tag{1}$$

PROOF. For each $x, y \in \mathscr{H}$ we have

$$(x, y)' = \langle x, y\rangle_0 + i\langle x, u'y\rangle_0. \tag{2}$$

But

$$\begin{aligned}\langle x, u'y\rangle_0 = \langle x, quq^\sigma y\rangle_0 &= \mathrm{Re}[\langle x, uq^\sigma y\rangle q^\sigma] \\ &= \mathrm{Re}[q^\sigma\langle x, uq^\sigma y\rangle] = \langle q^\sigma x, q^\sigma y\rangle_0.\end{aligned}$$

Also, using the fact that

$$\mathrm{Re}(a\, b\, c) = \mathrm{Re}(b\, c\, a) = \mathrm{Re}(c\, a\, b), \qquad \forall a, b, c \in \mathscr{Q},$$

we have $\langle x, y\rangle_0=\mathrm{Re}[\langle x, y\rangle q\, q^\sigma]=\mathrm{Re}[q^\sigma\langle x, y\rangle q]=\langle q^\sigma x, q^\sigma y\rangle_0$. Substitution in (2) gives (1).

This result has not appeared in the literature to date.

It is also possible to give $\mathscr{H}$ the structure of a real Hilbert space $\mathscr{H}_0$. The real-bilinear form $\langle\ ,\ \rangle_0$ on $\mathscr{H}$ defined by $\langle x, y\rangle_0=\mathrm{Re}\langle x, y\rangle$, $\forall x, y \in \mathscr{H}$, satisfies the axioms for an inner product on a real Hilbert space.

If S is an orthonormal basis for $\mathscr{H}$, then the set $S_0=\{i_\mu y : y \in S,\ \mu=0, 1, 2, 3\}$ is an orthonormal basis for $\mathscr{H}_0$.

4. **Classes of operators.** The fact that it is possible to impose real and complex Hilbert space structures on the underlying set of a quaternionic Hilbert space makes it possible to assign more than one meaning to the adjective "linear".

DEFINITION 1. Let $\mathscr{H}$ and $\mathscr{K}$ be quaternionic Hilbert spaces and let $A:\mathscr{H}\to\mathscr{K}$ be an operator such that

$$A(x + y) = Ax + Ay, \qquad \forall x, y \in \mathscr{H}. \tag{3}$$

A is called *real-linear* or ***R*-*linear***, if for each real quaternion q,

$$A(qx) = qAx, \qquad \forall x \in \mathscr{H}. \tag{4}$$

A is called $\boldsymbol{C}(i_1)$*-linear* if (4) holds for each q in $\boldsymbol{C}(i_1)$, *linear* if (4) holds for every quaternion q.

The classes of continuous real-linear, continuous $C(i_1)$-linear, and continuous linear operators from $\mathscr{H}$ to $\mathscr{K}$ are denoted by $\mathscr{L}(\mathscr{H}_0, \mathscr{K}_0)$, $\mathscr{L}(\mathscr{H}_1(i_1), \mathscr{K}_1(i_1))$, and $\mathscr{L}(\mathscr{H}, \mathscr{K})$ respectively; in the case where $\mathscr{H}=\mathscr{K}$ the abbreviations $\mathscr{L}(\mathscr{H}_0)$, $\mathscr{L}(\mathscr{H}_1(i_1))$, and $\mathscr{L}(\mathscr{H})$ are used.

DEFINITION 2. An operator $A:\mathscr{H}\to\mathscr{K}$ is called *colinear* if (3) holds and there is a fixed automorphism ω of $\mathscr{Q}$ such that, for each quaternion q, $A(qx)=q^{\omega}Ax$, $\forall x\in\mathscr{H}$.

When we wish to emphasise the particular automorphism ω associated with A we say that A is *ω-linear*. The class of continuous ω-linear operators from $\mathscr{H}$ to $\mathscr{K}$ is denoted by $\mathscr{L}_{\omega}(\mathscr{H},\mathscr{K})$.

It is well known (see, for example, [**1**]) that every colinear operator is a quaternion multiple of a linear operator. More precisely, given any non-zero quaternion a, let $\omega(a)$ denote the automorphism $\omega(a):q\mapsto aqa^{-1}$ of $\mathscr{Q}$. Then

$$\mathscr{L}_{\omega(a)}(\mathscr{H},\mathscr{K})=\{aT: T\in\mathscr{L}(\mathscr{H},\mathscr{K})\}. \tag{5}$$

DEFINITION 3. The *adjoint* of a continuous real-linear operator A: $\mathscr{H}\to\mathscr{H}$ is the unique continuous real-linear operator $A^*:\mathscr{H}\to\mathscr{H}$ such that $\langle Ax, y\rangle_0=\langle x, A^*y\rangle_0$, $\forall x, y\in\mathscr{H}$.

This definition is justified by the Riesz representation theorem for real Hilbert space. If A is $C(i_1)$-linear, then A^* is just the complex Hilbert space adjoint. If A is linear, the Riesz representation theorem for quaternionic Hilbert space can be used to show that $\langle Ax, y\rangle=\langle x, A^*y\rangle$, $\forall x, y\in\mathscr{H}$. It is easily verified that the mapping $A\mapsto A^*$ is a real-linear involution of $\mathscr{L}(\mathscr{H}_0)$ with the property:

$$(AB)^*=B^*A^*, \qquad \forall A, B\in\mathscr{L}(\mathscr{H}_0).$$

It should also be noted that, for any quaternion a, the adjoint of the multiplication operator aI is $a^{\sigma}I$.

5. **Characterisation of real-linear operators.** The basic technique is to carry over to operators the decomposition formula given in the following lemma.

LEMMA. *Let (i_1, i_2, i_3) be any right-handed orthonormal triad of vector quaternions. Then the components q_0, q_1, q_2, q_3 of an arbitrary quaternion $q=\sum_{\nu=0}^{3} q_\nu i_\nu$ can be found from the matrix expression*

$$[q_0, q_1, q_2, q_3]^T=Q[q, -qi_1, -qi_2, -qi_3]^T,$$
$$\text{where } 4Q=\begin{bmatrix} 1 & i_1 & i_2 & i_3\\ -i_1 & 1 & i_3 & -i_2\\ -i_2 & -i_3 & 1 & i_1\\ -i_3 & i_2 & -i_1 & 1\end{bmatrix} \tag{6}$$

PROOF. The 1×4 quaternion matrix $[q\ -qi_1\ -qi_2\ -qi_3]^T$ can be written in the form

$$[q, -qi_1, -qi_2, -qi_3]^T=4Q[q_0, q_1, q_2, q_3]^T.$$

But it can be shown by direct matrix multiplication that $(4Q)^2=4I$. Thus

$$[q_0, q_1, q_2, q_3]^T = Q[q, -qi_1, -qi_2, -qi_3]^T = Qq[1, -i_1, -i_2, -i_3]^T.$$

This *canonical decomposition* formula for quaternions is the inspiration for the main result.

THEOREM 2. *Every continuous real-linear operator A mapping a quaternionic Hilbert space $\mathscr{H}$ into a quaternionic Hilbert space $\mathscr{K}$ can be expanded in the form of an arbitrary right-handed orthonormal triad (i_1, i_2, i_3) of vector quaternions in the form:*

$$A = A_0 + i_1A_1 + i_2A_2 + i_3A_3, \tag{7}$$

where A_0, A_1, A_2, A_3 are continuous linear operators mapping $\mathscr{H}$ into $\mathscr{K}$. Moreover, they are uniquely determined by A and the choice of quaternion basis (i_1, i_2, i_3).

PROOF. The proof is based on a computation which imitates that used to prove the lemma. Let $A:\mathscr{H}\to\mathscr{K}$ be a continuous real-linear operator and let (i_1, i_2, i_3) be a right-handed orthonormal triad of vector quaternions. We define the operators A_0, A_1, A_2, A_3 by the "operator matrix equation"

$$[A_0, A_1, A_2, A_3]^T = QA[I, -i_1I, -i_2I, -i_3I]^T, \tag{8}$$

where I denotes the identity operator on $\mathscr{H}$. In other words:

$$\begin{aligned}
4A_0 &= A - i_1Ai_1I - i_2Ai_2I - i_3Ai_3I,\\
4A_1 &= -i_1A - Ai_1I - i_3Ai_2I + i_2Ai_3I,\\
4A_2 &= -i_2A + i_3Ai_1I - Ai_2I - i_1Ai_3I,\\
4A_3 &= -i_3A - i_2Ai_1I + i_1Ai_2I - Ai_3I.
\end{aligned}$$

Multiplying (8) on the left by Q gives:

$$A[I, -i_1I, -i_2I, -i_3I]^T = 4Q[A_0, A_1, A_2, A_3]^T, \tag{9}$$

the first row of which is (7). Clearly A_0, A_1, A_2, A_3 are continuous, and routine computations show that they are linear.

To prove uniqueness, suppose that $A=A_0+i_1A_1+i_2A_2+i_3A_3=B_0+i_1B_1+i_2B_2+i_3B_3$, where A_0, A_1, A_2, A_3 are the linear operators defined by (8), and B_0, B_1, B_2, B_3 are also linear operators. A simple calculation shows that

$$A[I, -i_1I, -i_2I, -i_3I]^T = 4Q[B_0, B_1, B_2, B_3]^T. \tag{10}$$

Combining (9) and (10) and recalling that the quaternion matrix Q is non-singular, we have

$$[A_0, A_1, A_2, A_3]^T = [B_0, B_1, B_2, B_3]^T \quad \text{as required.}$$

The main result of this theorem can be expressed concisely by $\mathscr{L}(\mathscr{H}_0, \mathscr{K}_0)=\{\sum_{\nu=0}^3 i_\nu A_\nu : A_\nu \in \mathscr{L}(\mathscr{H}, \mathscr{K})\}$. The expression $\sum_{\nu=0}^3 i_\nu A_\nu$ is called the *canonical decomposition* of the operator. In the case where $\mathscr{H}=\mathscr{K}$, we see that $\mathscr{L}(\mathscr{H})$ is a real algebra and $\mathscr{L}(\mathscr{H}_0)$ is the quaternion algebra generated by it.

COROLLARY. $\mathscr{L}(\mathscr{H}_1(i_1), \mathscr{K}_1(i_1))=\{A_0+i_1A_1 : A_0, A_1 \in \mathscr{L}(\mathscr{H}, \mathscr{K})\}$.

PROOF. If A is $C(i_1)$-linear, then (8) gives $A_2=A_3=0$.

Comparison of this result with (5) reveals that the $\omega(i_1)$-linear operators form a proper subset of $\mathscr{L}(\mathscr{H}_1(i_1), \mathscr{K}_1(i_1))$.

We also observe that if $A:\mathscr{H}\to\mathscr{H}$ is a continuous real-linear operator, then the adjoint A^* of A is given by

$$A^* = A_0^* - i_1A_1^* - i_2A_2^* - i_3A_3^*.$$

Thus A is Hermitian if and only if A_0 is Hermitian and A_1, A_2, A_3 are skew-Hermitian.

It now remains to investigate the effect on A_0, A_1, A_2, A_3 of a rotation of the triad (i_1, i_2, i_3).

THEOREM 3. *Let (i_1, i_2, i_3) and (i_1', i_2', i_3') be two right-handed orthonormal triads of vector quaternions and let $L=[l_{\mu\nu}]$ be the matrix of the three-dimensional rotation which transforms the first triad into the second. Let $A:\mathscr{H}\to\mathscr{K}$ be a continuous real-linear operator and write*

$$A = A_0 + i_1A_1 + i_2A_2 + i_3A_3 = A_0' + i_1'A_1' + i_2'A_2' + i_3'A_3'.$$

Then

$$[A_0', A_1', A_2', A_3'] = [A_0, A_1, A_2, A_3]\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & L \end{array}\right].$$

PROOF. By definition of L, we have

$$[i_1'\ i_2'\ i_3'] = [i_1\ i_2\ i_3]L. \tag{11}$$

Equivalently, since L is a real orthogonal matrix, $[i_1\ i_2\ i_3]=[i_1'\ i_2'\ i_3']L^T$.

$$A = A_0 + \sum_{\mu=1}^3 i_\mu A_\mu = A_0 + \sum_{\mu=1}^3 \sum_{\nu=1}^3 l_{\mu\nu} i_\nu' A_\mu.$$

But $A=A_0'+\sum_{\nu=1}^{3} i_\nu' A_\nu'$. So, by the uniqueness of the canonical decomposition, $A_0'=A_0$ and $A_\nu'=\sum_{\mu=1}^{3} l_{\mu\nu}A_\mu$ ($\nu=1, 2, 3$).

As immediate consequences of this theorem we have

COROLLARY 1. *Let* $A=A_0+i_1A_1+i_2A_2+i_3A_3=A_0'+i_1'A_1'+i_2'A_2'+i_3'A_3'$, *as in Theorem* 3. *Then* $A_0=A_0'$. *Furthermore, if say* $i_1=i_1'$, *then* $A_1=A_1'$.

COROLLARY 2. *Suppose that the operator* $A=A_0+i_1A_1+i_2A_2+i_3A_3$ *is* $C(u)$*-linear for some unit vector quaternion* u. *Then the operators* A_1, A_2, A_3 *are all real multiples of the same linear operator. Equivalently,* $A=A_0+uA_u$, *where* A_u *is a uniquely determined linear operator.*

PROOF. Choose a right-handed orthonormal triad (i_1', i_2', i_3') of vector quaternions such that $i_1'=u$. Then we can write $A=A_0+i_1'A_u$. By Corollary 1, A_u does not depend on the choice of i_2' and i_3'. Let L be the matrix defined by (11). Then $[A_1, A_2, A_3]=[A_u, 0, 0]L^T$, i.e. $A_1=l_{11}\,A_u$, $A_2=l_{21}\,A_u$, $A_3=l_{31}\,A_u$.

EXAMPLE. We illustrate our results with a brief discussion of matrix operators. Let $P=[p_{jk}]$ be any $n\times n$ matrix with quaternion entries and let P_L be the operator defined on the n-dimensional quaternionic Hilbert space $\mathcal{Q}^n$ by multiplying the column vectors of $\mathcal{Q}^n$ on the left by P—i.e., $y=P_Lx$ is given by

$$y_j = \sum_{k=1}^{n} p_{jk}x_k \qquad (j = 1, \cdots, n).$$

The operator P_L is linear if all the elements of P are real, but in general we can only say that P_L is a continuous real-linear operator. The canonical decomposition of P_L is given by

$$P_L = (P_0)_L + i_1(P_1)_L + i_2(P_2)_L + i_3(P_3)_L,$$

where P_0, P_1, P_2, P_3 are the real matrices obtained by decomposing the entries in P. The adjoint of P_L is $(P^*)_L$, where P^* is the transposed conjugate matrix $[(p_{kj})^\sigma]$.

It should be mentioned that the matrix P can be made to operate on $\mathcal{Q}^n$ in another way—by multiplying the row vectors x^T on the right by the transposed matrix P^T. In other words, we define the operator P_R on $\mathcal{Q}^n$ by writing $y=P_Rx$, where $y_j=\sum_{k=1}^{n} x_kp_{jk}$ ($j=1, \cdots, n$). The operator P_R is linear and is in fact the "orthodox" action on $\mathcal{Q}^n$ of a matrix with quaternion entries. That is, up to the present every discussion of matrix operators has used right multiplication by a matrix or, equivalently, has used left multiplication in the context of a right $\mathcal{Q}$-module (see, for example, **[2]**). Thus the resulting operators have always been linear.

The possibility of constructing a functional calculus for a class of real-linear operators will be discussed in a subsequent paper.

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