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ON THE SHAPE OF TORUS-LIKE CONTINUA AND COMPACT CONNECTED TOPOLOGICAL GROUPS[1]

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ABSTRACT. In this paper it is shown that if X is a torus-like continuum, then X has the shape of a compact connected abelian topological group. Let Π be a collection of compact connected Lie groups. In light of the above result it is natural to ask if a Π-like continuum has the shape of a compact connected topological group. An example is given to show that this is not the case.

Introduction. Let C denote the category of compact Hausdorff spaces and continuous maps and let $H: C \to HC$ be the homotopy functor. Let $S: C \to SC$ be the functor of shape in the sense of Holsztyński for the projection functor H [5]. It is assumed that the reader is familiar with the equivalence of this approach to shape with that of Mardesič and Segal using ANR-systems [12]. A precise statement of this equivalence with a proof is given in the Appendix of [6]. In this paper we give the shape classification of all torus-like continua. If X is a torus-like continuum, then it has the same shape as a compact connected abelian topological group. It is shown, in fact, that X has the same shape as char $H^1(X)$ where $H^n(X)$ is n-dimensional Čech cohomology over the integers. Using our knowledge of the shape properties of compact connected abelian topological groups contained in [6], [7], and [8] several properties of torus-like continua are derived.

In light of the above result about torus-like continua, it is natural to ask if a Π-like continuum might not have the shape of a compact connected topological group where Π is a collection of compact connected Lie groups. An example is given to show that this is not the case. In this section of the paper we also show that if G is a compact connected topological group with $H^1(G)=0$, then $H^n(G)/\mathrm{Tor}\, H^n(G)$ has property L for all $n \geq 0$. This result is invoked to show that the above example cannot have the shape of a compact connected topological group. An example is

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also given of a nonmovable G with $H^1(G)=Z$. In [6] it was shown that for A a compact connected abelian topological group, A is movable if and only if $H^1(A)$ has property L. The last example shows that the fact that A is assumed to be abelian is not superfluous in that theorem.

Throughout the paper we let $H^n(X)$ denote n-dimensional Čech cohomology with integer coefficients. We assume the notation of [6], [7], and [8].

1. **The shape of torus-like continua.** In this section we show that if X is a torus-like continuum, then X has the shape of a compact connected abelian topological group and we draw some corollaries from this. First we recall the definition of Π-like.

1.1. DEFINITION. Let Π be a collection of polyhedra. Then a continuum X is said to be Π-*like* provided that, for every open cover $\mathcal{U}$ of X, there is a $P\in\Pi$ and a continuous $f(X)=P$ such that for $p\in P, f^{-1}(p)\subset U$ for some $U\in\mathcal{U}$. If Π is the collection of tori, $\Pi=\{T^n:n\geqq 0\}$, then the continuum is said to be *torus-like*.

1.2. THEOREM. *Let X be a torus-like continuum. Then X has the shape of a compact connected abelian topological group.*

PROOF. First we suppose that X is metrizable. In this case there is an inverse sequence of tori $\mathbf{X}=\{T_i^{n_i}; \pi_{ij}; i\leqq j<\omega_0\}$ which has X as inverse limit [10]. Then $\mathbf{X}$ is an ANR-sequence associated with X in the terminology of [12]. Let π_{ij}^* be the unique continuous homomorphism homotopic to π_{ij} by [14] and let A be the inverse limit of $\mathbf{A}=\{T_i^{n_i}; \pi_{ij}^*; i\leqq j<\omega_0\}$. Then A is a compact connected abelian topological group and A is associated with the ANR-sequence $\mathbf{A}$. We define a map of ANR-systems $\mathbf{f}:\mathbf{X}\to\mathbf{A}$ by $f(i)=i$ and $f_i=1:T_i^{n_i}\to T_i^{n_i}$. Then $\mathbf{f}$ is a homotopy equivalence between the ANR-systems $\mathbf{X}$ and $\mathbf{A}$. Thus X and A have the same shape (see the Appendix of [6] for the equivalence of the approach to shape in [12] to that in [5]). Now we will proceed to prove the theorem in general. First note that if X and Y are metrizable and torus-like and $F\in \mathrm{Mor}_{SC}(X, A_X)$ and $G\in\mathrm{Mor}_{SC}(A_Y, Y)$ are shape equivalences with A_X and A_Y compact connected abelian topological groups, then if we define $P:\mathrm{Hom}(A_X, A_Y)\to\mathrm{Mor}_{SC}(X, Y)$ by $P(h)=G\circ S(h)\circ F$, then P is one-to-one and onto by Theorem 1.2 of [6].

Suppose now that X is a torus-like continuum which is not metrizable. Then by [9], there is an inverse system $\{X_\alpha; \pi_{\alpha\beta}; \alpha\leqq\beta\in A\}$ having X as its inverse limit such that each X_α is metrizable and torus-like. For each $\alpha\in A$, let A_α be a compact connected abelian topological group and $F_\alpha\in\mathrm{Mor}_{SC}(X_\alpha, A_\alpha)$ and $G_\alpha\in\mathrm{Mor}_{SC}(A_\alpha, X_\alpha)$ such that $F_\alpha\circ G_\alpha$ is the identity shape morphism on A_α and $G_\alpha\circ F_\alpha$ is the identity shape morphism on X_α. Then for each $\alpha \leqq \beta$, let $\pi^*_{\alpha\beta}$ be the unique continuous homomorphism from A_β to A_α such that $S(\pi_{\alpha\beta})=G_\beta \circ S(\pi^*_{\alpha\beta}) \circ F_\alpha$ as above. Then $\{A_\alpha; \pi^*_{\alpha\beta}; \alpha \leqq \beta \in A\}$ becomes an inverse system of compact connected abelian topological groups. Let A be the inverse limit of this inverse system. Then A is a compact connected abelian topological group. It now follows easily from the continuity of the shape functor [5] that $S(A)=S(X)$ and the proof is complete.

Using the results of [6], [7], and [8] we now give some immediate consequences of Theorem 1.2.

1.3. Corollary. *If X is a torus-like continuum, then X has the same shape as* char $H^1(X)$.

Proof. By Theorem 1.2 there is a compact connected abelian topological group A which is shape equivalent to X. This implies that $H^1(X) \simeq H^1(A)$. Let $A_X = \text{char } H^1(X)$. Then $H^1(X) \simeq H^1(A_X)$ by Corollary 1.5 of [8]. Thus $H^1(A) \simeq H^1(A_X)$. By Theorem 1.4 of [8], char $A \simeq$ char A_X. This implies that $A \simeq A_X$ and that X, A, and A_X all have the same shape.

1.4. Corollary. *Let X be a continuum and Y a torus-like continuum. Then* $\text{Mor}_{SC}(X, Y)$ *is in one-to-one correspondence with* $\text{Hom}(H^1(Y), H^1(X))$ *by the function which takes* $F \in \text{Mor}_{SC}(X, Y)$ *to the homomorphism* F^*.

Proof. By Theorem 1.7 of [8] this is true for Y a compact connected abelian topological group and since Y has the same shape as a compact connected abelian topological group, the corollary follows.

1.5. Corollary. *Let X be a torus-like continuum. Then X is movable if and only if $H^1(X)$ has property* L.

Proof. This is because this is true for compact connected abelian topological groups by Theorem 2.5 of [6] and Theorem 1.4 of [8].

Of course, the Čech cohomology of a torus-like continuum Y will be the same as its associated compact connected abelian topological group. Thus one can apply the results of [3] to compute the Čech cohomology of Y over any integral domain R.

In the next section we will show that if Π is a collection of compact connected Lie groups, then a continuum X may be Π-like without having the shape of a compact connected topological group.

2. **Compact connected topological groups.** In this section we use a well-known structure theorem for compact connected topological groups to derive a shape invariant property for such spaces. This is used to show that a certain continuum cannot have the shape of a compact connected topological group.

2.1. THEOREM. *Let G be a compact connected topological group. Then G is isomorphic to a group $(A\times B)/D$ where A is a product of simple, connected, simply connected compact Lie groups; B is a compact connected abelian topological group; and D is a totally disconnected closed central subgroup of $A\times B$.*

This is essentially 6.59 on p. 75 in [**2**]. Now we recall the definition of property L.

2.2. DEFINITION. Let H be an abelian group and J a subgroup of G. Then J is said to *admit division* if whenever $h\in H$ and n is a positive integer, then $nh\in J$ implies that $h\in J$. This is equivalent to saying that H/J is torsion free. The group H is said to have *property* L if every finite subset of H is contained in a finitely generated subgroup that admits division.

In [**8**] it was shown that if X is a movable continuum, then $H^n(X)/\text{Tor } H^n(X)$ has property L for all $n\geqq 0$. This fact will be used in the proof of the next theorem. We also note that if $X=\prod_{\alpha\in A} X_\alpha$ where each X_α is an ANR, then X is movable. This can be seen by letting X be the limit of the finite subproducts $\prod_{i=1}^n X_{\alpha_i}$ with the bonding maps just projections onto subproducts. This is clearly a movable ANR-system associated with X.

2.3. THEOREM. *Let G be a compact connected topological group. Then if $H^1(G)=0$, then $H^n(G)/\text{Tor } H^n(G)$ has property* L *for all $n\geqq 0$.*

PROOF. Let $G=(A\times B)/D$ as in Theorem 2.1 and suppose that $H^1(G)=0$. Then suppose that $B\neq 0$. Then let $\pi_B:A\times B\to B$ be the projection map. Now π_B is a continuous homomorphism. Therefore $\pi_B|D$ is a continuous homomorphism and thus $\pi_B|D: D\to\pi_B(D)$ is open and thus $\pi_B(D)$ is also 0-dimensional since $\pi_B|D$ is open and closed. This implies that $B/\pi_B(D)\neq 0$. Let H be the commutator subgroup of G, then $G/H\simeq B/\pi_B(D)\neq 0$. By Theorem 3.1 of [**8**], $H^1(G/H)\simeq H^1(G)\neq 0$, a contradiction. That is, if $H^1(G)=0$, then $B=0$. Thus $G\simeq A/D$. Now $A=\prod_{\gamma\in\Gamma} L_\gamma$ where each L_γ is a simple, simply connected, compact connected Lie group. Since each L_γ is an ANR, A is movable. Thus $H^n(A)/\text{Tor } H^n(A)$ has property L for all $n\geqq 0$ [**8**, Theorem 4.4]. By [**4**, 3.16, p. 330], the quotient homomorphism $p:A\to A/D=G$ induces an injection $p^*:H^n(G)/\text{Tor } H^n(G)\to H^n(A)/\text{Tor } H^n(A)$. Now a subgroup of a group having property L has property L. Thus $H^n(G)/\text{Tor } H^n(G)$ has property L for all $n\geqq 0$.

2.4. EXAMPLE. Here we will show that if X is a Π-like continuum where Π is a collection of compact connected Lie groups, then X need not have the shape of a compact connected topological group. Let Σ_a be a solenoid which is not a circle. Then $H^1(\Sigma_a)=\text{char}\,\Sigma_a$ does not have property L and is torsion free. Let $X=\Sigma^2\Sigma_a$ be the two-fold suspension of Σ_a. Then X is a 3-sphere-like continuum with the 3-sphere a compact connected Lie group. However, $H^3(X)\simeq H^1(\Sigma_a)$ is torsion free and does not have property L. Also, $H^1(X)=0$. Thus X cannot have the shape of a compact connected topological group.

In [6] it was shown that if A is a compact connected abelian topological group, then A is movable if and only if $H^1(A)$ has property L. We will now give an example of a compact connected topological group G which is movable with $H^1(G)=Z$. This shows that this theorem is not true for compact connected topological groups.

2.5. EXAMPLE. Recall that the center of $SU(k)=L_k$ is just the cyclic group Z_k [1, p. 31]. Let $\{p_i\}$ be a sequence of increasing prime numbers. Observe that one can imbed $Z_{p_1\cdots p_n}\subset Z_{p_1}\times\cdots\times Z_{p_n}\subset L_{p_1}\times\cdots\times L_{p_n}$ by the homomorphism $1\mapsto(1, 1, \cdots, 1)$. One can also imbed $Z_{p_1\cdots p_n}$ in the circle group T as the $(p_1\cdots p_n)$th roots of unity. Consider $Z_{p_1\cdots p_n}\times Z_{p_1\cdots p_n}\subset T\times[L_{p_1}\times\cdots\times L_{p_n}]$ as the product of these two imbeddings. Then consider $\Delta:Z_{p_1\cdots p_n}\to Z_{p_1\cdots p_n}\times Z_{p_1\cdots p_n}$ as the diagonal map and let $D_n=\Delta(Z_{p_1\cdots p_n})\subset Z_{p_1\cdots p_n}\times Z_{p_1\cdots p_n}\subset T\times[L_{p_1}\times\cdots\times L_{p_n}]$. Let $G_n=[T\times(L_{p_1}\times\cdots\times L_{p_n})]/D_n$. Let $\psi_{nm}:T\times(L_{p_1}\times\cdots\times L_{p_m})\to T\times(L_{p_1}\times\cdots\times L_{p_n})$ be defined by $\psi_{nm}|T$ is equivalent to the complex map $z\mapsto z^{p_{n+1}\cdots p_m}$ and $\psi_{nm}(L_{p_1}\times\cdots\times L_{p_m})=L_{p_1}\times\cdots\times L_{p_n}$ is just the projection onto the first n factors. Note that $\psi_{nm}(D_m)=D_n$. Thus ψ_{nm} induces a continuous homomorphism $\pi_{nm}:G_m\to G_n$. Clearly $\pi_{nm}\circ\pi_{mp}=\pi_{np}$ for $n\leqq m\leqq p$. Thus we have an inverse sequence of compact connected Lie groups $\{G_n;\ \pi_{nm};\ n\leqq m<\omega_0\}$. Let G be the inverse limit group. Then we claim that $H^1(G)=Z$ and that G is nonmovable.

First we show that G is nonmovable. This is equivalent to showing that the ANR-sequence $\{G_n;\ \pi_{nm};\ n\leqq m<\omega_0\}$ is not movable. Note that the fundamental group of G_n is $\pi_1(G_n)=Z\times Z_{p_1\cdots p_n}$ and $\pi^*_{nm}:\pi_1(G_m)\to\pi_1(G_n)$ is the homomorphism taking $(1, 0)$ to $(p_{n+1}\cdots p_m, 0)$ and $(0, 1)$ to $(0, 1)$. Let $n=1$. Then in the definition of movability one must be able to find an $m\geqq 1$ such that for all $p\geqq m$, there is a continuous map r^{mp}: $G_m\to G_p$ such that π_{1m} is homotopic to $\pi_{1p}\circ r^{mp}$. However, this would insure that there is a homomorphism $h^{mp}:\pi_1(G_m)\to\pi_1(G_p)$ such that $\pi^*_{1m}=\pi^*_{1p}\circ h^{mp}$. However, letting $p=m+1$, no such h^{mp} can exist. Thus G cannot be movable.

To show that $H^1(G)=Z$ note that if Q is the commutator subgroup of G, then the quotient homomorphism $p:G\to G/Q$ induces an isomorphism $p^*:H^1(G/Q)\simeq H^1(G)$ by Theorem 3.1 of [8]. But G/Q is the inverse limit of G_n/Q_n where Q_n is the commutator subgroup of G_n. Now $Q_n=L_{p_1}\times\cdots\times L_{p_n}$ and $G_n/Q_n\simeq T$ for all n. Now the induced homomorphism from $G_m/Q_m \to G_n/Q_n$ is an isomorphism. Thus the inverse limit of G_n/Q_n is just T and $G/Q \simeq T$. Thus $H^1(G/Q) \simeq H^1(G) \simeq Z$.

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