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MORE NONEUCLIDIAN PID'S AND DEDEKIND DOMAINS WITH PRESCRIBED CLASS GROUP

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ABSTRACT. Let Z denote the integers, Q the rationals, X an indeterminate and G a finitely generated abelian group. Then there is a Dedekind domain D such that $Z[X]\subset D\subsetneqq Q[X]$, and D has class group G. If $G=0$ then D is a noneuclidian PID.

If D is a Dedekind domain, then the nonzero ideals of D form an abelian group I under multiplication. An obvious subgroup is J, the group of principal ideals. The group I/J is called the *class group* or *divisor class group* of D. In [C] Claborn gives a method for constructing a Dedekind domain with any pre-assigned abelian group as class group. We offer here a theorem which allows one, given a finitely generated abelian group G, to construct a Dedekind domain with G as class group. We feel that our construction offers much simpler rings than Claborn's and that their number theory is much more tractable. As evidence of this, it is easy to see that the PID's constructed here (i.e., those with class group zero) are not euclidian rings.

We refer the reader to the first few pages of Professor Samuel's Tata notes [S_2] or to Bourbaki [B] for elegant exposition on the class group. Our methods are extracted from these sources, [S_1] and [AEH].

In order to avoid repetition, all valuation rings are assumed to be rank one discrete. By an *algebraic extension* of a valuation ring V we mean a valuation ring V^* which extends V and whose residue field is algebraic over that of V.

THEOREM. *Let K be a field and $K(t)$ a simple transcendental extension of K. Let $V_1, \cdots, V_n$ be distinct, rank one discrete valuation rings with quotient field K. For each i let $\{V^*_{i,j}\}_{j=1}^{g_i}$ be a finite collection of distinct, algebraic extensions of V_i to $K(t)$. Let $e_{i,j}$ denote the ramification index of*

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$V^*_{i,j}$ over V_i and set

$$D = \left(\bigcap_{i,j} V^*_{i,j}\right) \cap K[t].$$

Then D is a Dedekind domain.

Set $m=\sum_i g_i$ and let G denote the free abelian group of rank m on the generators $P_{i,j}$. Let L denote the subgroup of G generated by the elements $R_i=\sum_{j=1}^{g_i} e_{i,j}P_{i,j}$ for $i=1, \cdots, n$. Then $C=G/L$ is the class group of D. Moreover, whenever $C=0$, D is a noneuclidian principal ideal domain.

PROOF. Let π_i be a prime element of V_i and $s=\prod \pi_i$. Then $D[1/s]=K[t]$. Obviously the essential valuations of s are exactly the $V^*_{i,j}$. Let $\Psi_{i,j}$ denote the center of $V^*_{i,j}$ on D. To see that D is Dedekind, it suffices to show that the $\Psi_{i,j}$ are maximal ideals of D since only maximal ideals containing s are lost when one goes from D to $D[1/s]$. The fact that the $\Psi_{i,j}$ are maximal follows from the assumption that the $V^*_{i,j}$ are algebraic extensions of the V_i. This implies that $D/\Psi_{i,j}$ is algebraic over the field V_i/π_i which makes it a field. To get Dedekind, one can also apply directly the result (5.7) of **[AEH]**.

Let $\mathscr{D}$ be the group of divisors of D and $\mathscr{P}$ the subgroup of principal divisors. Let $P_{i,j}\in\mathscr{D}$ be the divisor corresponding to $\Psi_{i,j}$, and let G be the subgroup of $\mathscr{D}$ generated by $\{P_{i,j}\}$. Since $D[1/s]=K[t]$ is a PID, it follows that $\mathscr{P}+G=\mathscr{D}$. Thus the class group $C=\mathscr{D}/\mathscr{P}=(G+\mathscr{P})/\mathscr{P}=G/\mathscr{P}\cap G$. Let $R_i=\sum_{j=1}^{g_i} e_{i,j}P_{i,j}$ and let L be the subgroup generated by the R_i. We claim $L=\mathscr{P}\cap G$. Containment one way is clear since R_i is the divisor of π_i. Conversely suppose $R=\sum n_{i,j}P_{i,j}\in\mathscr{P}$. This asserts that there is an $\alpha\in K(t)$ whose divisor is precisely R. But it must then be true that $\alpha\in K$ since only elements of K can have value zero in all of the other essential valuations of D (the f-adic valuations where f is an irreducible member of $K[t]$). So if v_i is the associated normed valuation of V_i and $v_i(\alpha_i)=r_i$, then $R=\sum r_iR_i$. Hence $C=G/L$ which is exactly what was to be seen.

The fact that D is not euclidian follows quickly from the transfinite construction on p. 289 of [S_1]. Were D euclidian it would have to be a PID and one could, by the adjunction of the reciprocal of one element, reduce to the case $D=V^*\cap K[t]$ where V^* is an unramified, algebraic extension of $V=V^*\cap K$. If the residue field of V^* is a proper extension of that of V, one cannot even proceed beyond the first step in the construction. If V^* and V have the same residue field, one cannot proceed beyond the A_w step where w is the first infinite cardinal. In this case, A_w is easily seen to be V and the construction cannot advance past A_w: If p is any prime of D which might be in $A_{w+1}\backslash A_w$, then the mapping $D\to D/p$ induces an isomorphism on V. Since D/p is a field, $V\to D/p$ cannot be surjective.

Let V be a rank one discrete valuation ring with quotient field K and $K[t]$ a simple transcendental extension of $K[t]$. If V^* is an extension of V to $K[t]$, then V^* is easily seen to have an infinite number of conjugates under the action of the group of fractional linear transformations of $K[t]$ over K. Thus to produce an infinite supply of algebraic extensions of V of a given ramification index e, we need exhibit only one. Let Q denote the rationals and p a prime integer. Let Z_p be the p-adic valuation ring of Q and $\hat{Z}_p$ the p-adic integers. If t is chosen to be an element of $\hat{Z}_p$ which has p-adic value one and is transcendental over Q, then $V=\hat{Z}_p \cap Q[t]$ is an unramified, algebraic extension of Z_p to $Q[t]$. To get a ramified extension of a simple transcendental extension of Q, one can just extend V to $Q[t^{1/c}]$. We can thus state the following.

COROLLARY. *Let G be a finitely generated abelian group, Q the rationals, Z the integers, and X an indeterminate. Then there is a Dedekind domain D with class group G such that $Z[X] \subset D \subsetneqq Q[X]$.*

PROOF. By the fundamental theorem of finitely generated abelian groups, $G=Z^{(n)} \oplus Z/n_1 \oplus \cdots \oplus Z/n_e$ where $Z^{(n)}$ is the free abelian group of rank n and Z/n_i is the cyclic group of order n_i for $i=1, \cdots, e$.

Let $P_0, P_1, \cdots, P_e$ be distinct integral primes. Choose $\{V_{0,j}\}_{j=1}^{n_0+1}$ to be distinct, unramified algebraic extensions of Z_{p_0} to $Q[t]$. Then for each i from 1 to e choose V_i to be an algebraic extension of Z_{p_i} to $Q(t)$ of ramification index n_i. By our theorem, $D=Q[t] \cap (\bigcap_j V_{0,j}) \cap (\bigcap_{i=1}^{e} V_i)$ is the desired example.

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