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# ON SIMPLE INJECTIVE RINGS

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ABSTRACT. It is proved that a simple ring which is injective on both sides must be artinian. This answers a question asked by C. Faith in the negative.

In this note we will prove the following

THEOREM. *A ring is the full ring of* $(n, n)$*-matrices over a field iff the following three conditions hold simultaneously*:

(1) $R$ *is simple*,

(2) ${}_RR$ *is injective*,

(3) $R_R$ *is injective*.

The proof depends on a result of Utumi [5]: Under conditions (2) and (3) of the theorem $xy=1$ implies $yx=1$. (The converse of this theorem holds if $R$ is a regular ring, cf. [4].) First we recall a

LEMMA. *Let* $R$ *be a regular simple ring without primitive idempotents* $\neq 0$. *Then there are infinitely many pairwise orthogonal idempotents* $e_i$ *with* $e_iR \simeq e_kR$.

For a proof see [2] or [3].

PROPOSITION. *Let* ${}_RR$ *be injective, the left singular ideal* $Z({}_AA)=0$, $\{e_i, i \in N\}$ *pairwise orthogonal idempotents with* $Re_i \simeq Re_k$. *Then there exist* $a, b \in R$ *with* $ab=1$, $ba \neq 1$.

PROOF. Define $R$-homomorphisms $f, g$: $\bigoplus_{i=1}^{\infty} Re_i \to \bigoplus_{i=1}^{\infty} Re_i$

$$f\colon \begin{array}{l} e_{2n} \to e_n, \\ e_{2n+1} \to 0, \end{array} \qquad g\colon e_n \to e_{2n}, \qquad n = 1, 2, \cdots .$$

It is clear that $e_n(gf)=e_n$ for all $n$, but $e_{2n+1}(fg)=0$. From the lemma of Zorn $\{Re_i, i \in N\}$ may be imbedded in a maximal set $\{Re_k, k \in K\}$ of independent left ideals. Then $\bigoplus_{k\in K} Re_k \subset {}_RR$ essential.

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The definition $e_k f = e_k g = e_k$ for $k \in K$, $k \notin N$ extends $f$, $g$ to homomorphisms $\bigoplus_{k\in K} Re_k \to \bigoplus_{k\in K} Re_k$. ${}_R R$ injective and $Z({}_R R)=0$ implies: $f$, $g$ may be extended in one and only one way to a map ${}_R R \to {}_R R$, that is $f$, $g$ are right multiplications by elements $a, b \in R$.

PROOF OF THE THEOREM. It is enough to prove that the conditions are sufficient. From the theory of quotient rings it follows that $R$ is a regular ring. Assume that $R$ has no primitive idempotents$\neq 0$. Then from the lemma and the proposition there must exist $a$, $b$ with $ab=1$, $ba\neq 1$. This contradicts the above result of Utumi. Thus $R$ has primitive idempotents$\neq 0$ and $R$ is artinian because $R$ equals its socle.

REMARK. The theorem answers a question of C. Faith in the negative. [**1**, p. 130, Problem 16.] As the referee informs me the same question was answered by E. Roos in a comment communicated to the referee: A simple right injective ring need not be left injective. *Proof.* Let $R$ be any integral domain, not a right Ore domain, and let $S$ be its maximal right quotient ring. Then, $S$ is known to be a simple ring which is regular and right injective. Moreover, by Utumi's theorem, $S$ cannot be left injective, since otherwise $xy=1 \Rightarrow yx=1$. Now, any $y\neq 0$ in $R$ has a left inverse $x$ in $S$, so this would imply that every nonzero element of $R$ is a unit of $S$. This is impossible, since given $s\neq 0$ in $S$, there exists $a\neq 0$ in $R$ such that $b=sa\neq 0$ in $R$, and then $s=ba^{-1}$. This would imply $R$ is a right Ore domain with right quotient field $S$, contrary to the assumption.

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