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PLANAR FOURIER TRANSFORMS AND DIOPHANTINE APPROXIMATION

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ABSTRACT. The radial behavior of its Fourier-Stieltjes transform in R^2 is related to the modulus of continuity of a measure; in certain cases the Hausdorff dimension of an exceptional set of lines can be estimated. Converse results use the theory of Diophantine approximation established by Besicovitch and Jarník.

1. We shall study metrical properties of a measure $\mu \geqq 0$ of compact support in the plane R^2, relating these properties to the behavior of the Fourier-Stieltjes transform $\hat{\mu}$ on lines through (0, 0). The simplest of the metrical properties is the Lipschitz condition of order $\alpha: \mu \in \Lambda_\alpha$ shall mean that $\mu(S) \ll (\text{diam } S)^\alpha$ for all planar sets S. A set of lines through (0, 0) is said to have a property P, if that property is valid for the corresponding set of directions θ in $[0, 2\pi)$.

THEOREM 1. *If $\mu \in \Lambda_\alpha$, $1 < \alpha < 2$, then $\hat{\mu}$ tends to 0 along all lines, except a set of Hausdorff dimension at most $2-\alpha$. The upper bound $2-\alpha$ cannot be improved.*

THEOREM 2. *The transform $\hat{\mu}$ tends to 0 along almost all lines, provided $\iint |x-y|^{-1}\mu(dx)\mu(dy) < \infty$.*

Theorem 4 is a converse to this.

2. It is convenient to write transforms as $\hat{\mu}(r\xi)$, with $r \geqq 0$ and $|\xi| = 1$, and to write $d\xi$ for normalized arc-length on the circle $|\xi| = 1$. The Bessel function of order 0 is

$$J_0(u) \equiv \int \exp i(x, \xi)\, d\xi, \quad \text{if } |x| = u \geqq 0.$$

Concerning J_0 we need to know that it has successive integrals $\varphi_0 = J_0$, φ_1, and φ_2 each satisfying $|\varphi(u)| \ll (1+|u|)^{-1/2}$ on the real axis. This follows

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from the asymptotic expansion for J_0 up to $O(u^{-4})$; see [**4**, pp. 46–51] or [**3**, XIV–XV].

The following identity is basic, and is stated for a complex measure μ; $f(r)$ is a suitable function of r.

$$\iint f(r)\,|\hat{\mu}(r\xi)|^2\,dr\,d\xi = \int\cdots\int \exp ir(\xi, x-y)\mu(dx)\overline{\mu(dy)}\,dr\,d\xi$$
$$= \iiint f(r)J_0(r\,|x-y|)\mu(dx)\overline{\mu(dy)}\,dr.$$

The problems at hand require bounds in which $f(r)$ is the characteristic function of an interval, but $\int_R^{2R} J_0(rX)\,dr$ does not decrease rapidly enough to obtain the necessary bounds, when $\alpha \geqq 3/2$. The following device is borrowed from summability: let $F \in C^2(-\infty, \infty)$ vanish outside $(1/2, 5/2)$, let $F \geqq 1$ on $[1, 2]$ and $F \geqq 0$ everywhere.

LEMMA 1. $|\int F(R^{-1}r)J_0(rX)\,dr| \ll \min(R, R^{-3/2}X^{-5/2})$ *for all* $R>0$, $X>0$.

In fact, we can integrate by parts to obtain

$$R^{-2}X^{-2}\left|\int F''(R^{-1}r)\varphi_2(rX)\right| \ll R^{-2}X^{-2}\int_{R_2}^{3R} |rX|^{-1/2}\,dr.$$

This yields the bound $R^{-3/2}X^{-5/2}$, and the bound $\ll R$ follows from $|J_0| \leqq 1$.

To prove the first part of Theorem 1 we observe that

$$\int_R^{2R}\int |\hat{\mu}(r\xi)|^2\,dr\,d\xi \leqq \iint F(R^{-1}r)\,|\hat{\mu}(r\xi)|^2\,dr\,d\xi$$
$$= \iiint F(R^{-1}r)J_0(r\,|x-y|)\,dr\mu(dx)\overline{\mu(dy)}$$
$$\ll \iint \min(R, R^{-3/2}\,|x-y|^{-5/2})\mu(dx)\mu(dy).$$

Let $P(t)$ be the product measure of the set $\{|x-y| \leqq t\}$, so that $P(t) \ll t^\alpha$ when $\mu \in \Lambda_\alpha$. The last integral is then

$$\int \min(R, R^{-3/2}t^{-5/2})\,dP(t) \leqq RP(R^{-1}) + 3R^{-3/2}\int_{R^{-1}}^{\infty} t^{-7/2}P(t)\,dt$$
$$\ll R^{1-\alpha}, \quad \text{as } \alpha \leqq 2.$$

Now suppose $R>e$, and there is an r in $[R, 2R]$ such that $|\hat{\mu}(r\xi)|^2 > (\log R)^{-1}$. Since $\hat{\mu}$ has bounded partial derivatives in the plane, we have $\int_R^{2R} |\hat{\mu}(r\xi)|^2\,dr > c(\log R)^{-2}$, and in fact the same inequality holds on an arc about ξ of length $c(\log R)^{-1}R^{-1}$. Dividing the circumference $|\xi|=1$ into arcs I of length approximately R^{-1}, and using the estimate for

$\int_R^{2R}\int(\cdot)\,dr\,d\xi$ obtained above, we see that the number N_R of arcs I such that $\sup\{|\hat{\mu}(r\xi)|^2: R\leqq r\leqq 2R\}>(\log R)^{-1}$ somewhere in I, has order of magnitude $N_R\ll R^{2-\alpha}(\log R)^3$. Letting $R=1, 2, \cdots, 2^k, \cdots$, we see that $\hat{\mu}(r\xi)\ll(\log r)^{-1}$ except on a ξ-set of Hausdorff dimension $\leqq 2-\alpha$; this is the first assertion in Theorem 1.

To prove Theorem 2 we prove in fact that $\hat{\mu}$ is in L^2 along almost all lines. Indeed

$$\int_0^R\int|\hat{\mu}(r\xi)|^2\,dr\,d\xi=\int_0^R\iint J_0(r\,|x-y|)\mu(dx)\mu(dy)\,dr.$$

But J_0 has a bounded primitive, whence

$$\int_0^R\int|\hat{\mu}(r\xi)|^2\,dr\,d\xi\leqq C\iint|x-y|^{-1}\,\mu(dx)\mu(dy)<\infty.$$

4. We show that the set of exceptional directions can attain the dimension $2-\alpha$; when $\alpha=1$ we obtain a precise converse to Theorem 2. The symbol $\|u\|$ now denotes the distance between a real number u and the nearest integer; plane vectors are denoted X, Y and lowercase letters denote real numbers.

Henceforth $(N_k)_1^\infty$ is always a strictly increasing sequence of positive integers and $\varepsilon(k)>0$ a function of natural numbers, tending to 0. Given η in $(0, 1]$ the set $F=F(N,\varepsilon,\eta)$ is defined

$$\text{(I)}\qquad F:-1\leqq x\leqq 1,\qquad \|N_kx\|\leqq\varepsilon(k)N_k^{-\eta},\qquad 1\leqq k<\infty.$$

For an appropriate sequence N, $\dim F\geqq(1+\eta)^{-1}$. For example F can be a dyadic set, the set of all sums $\sum\pm 2^{-m_j}$ where $1\leqq m_1<\cdots<m_j<m_{j+1}<\cdots$ and $m_j\leqq(1+\eta)j+o(j)$. We can set $N_k=2^{m_j}$ provided $m_{j+1}\geqq(1+\eta)m_j+1-\log\varepsilon(k)/\log 2$; the existence of a sequence of 'exceptional' indices m_{j+1} is compatible with the first condition. Also, F is represented as a product set and thus carries a product measure, which is in Λ_c for all $c<(1+\eta)^{-1}$. (The Lipschitz condition can be established from elementary properties of binary expansions.) Hence $F\times F$ carries a product measure in each class Λ_{2c}, $c<(1+\eta)^{-1}$.

Let H be the set of real numbers t for which the Diophantine inequalities

$$\text{(II)}\qquad 1\leqq v\leqq\varepsilon^{-1/2}(k)N_k^\eta,\qquad \|vt\|\leqq\varepsilon^{1/3}(k)N_k^{-1}$$

have an integer solution for each $k=1, 2, \cdots$. For this integer v, and x in F,

$$\|vN_kx\|\leqq v\,\|N_kx\|\leqq\varepsilon^{1/2}(k),\qquad \|vN_ktx\|\leqq\|vtu\|+|vt|\,|N_kx-u|$$

for any integer u. Choosing $u=u(x)$ to be the integer closest to $N_k x$ we find

$$|u| \leqq 1 + N_k, \qquad |N_k x - u| \leqq \varepsilon(k) N_k^{-\eta},$$

whence

$$\|vN_k tx\| \leqq 2N_k \cdot \varepsilon^{1/3}(k) N_k^{-1} + |t|\,\varepsilon^{1/2}(k) = o(1).$$

By the method of Jarník ([**1**], [**2**], [**5**], [**6**]), when (N_k) is sufficiently sparse, dim $H=2(1+\eta^{-1})^{-1}=2\eta(1+\eta)^{-1}$. In order to apply the arguments cited, it is easier in fact to treat a smaller set, defined by stronger inequalities

$$\text{(II*)} \qquad \tfrac{1}{2}N_k^\eta \leqq v \leqq N_k^\eta, \qquad \|vt\| \leqq \varepsilon^{1/3}(k) N_k^{-1}.$$

(Jarník's method can be simplified somewhat by the use of Borel measures [**7**].) Setting $\alpha=2(1+\eta)^{-1}$, $2-\alpha=2\eta(1+\eta)^{-1}$ we can state

THEOREM 3. *There exists a probability measure σ on the line, in each class Λ_c, $c<2\alpha$, and a set H of dimension $2-\alpha$, such that for each t in H, there is a sequence $r_k(t)\to\infty$ with* $\lim \|r_k x\| = \lim \|r_k tx\| = 0$, *uniformly for all x in the closed support of σ.*

Theorem 3 complements Theorem 2 and establishes the exponent $2-\alpha$ as best-possible; for $\mu=\sigma\times\sigma$ belongs to Λ_c whenever $c<\alpha$, but $\hat\mu(2\pi r_k(t), 2\pi r_k(t)t)\to 1$ so that the exceptional set of directions includes H, and hence has dimension exactly $2-\alpha$.

The calculations in Theorem 3 can be applied when $\eta=1$; here Dirichlet's principle yields a solution of (II) for every real t, so that the transform $\hat\mu$ tends to zero along *no* line (the vertical line corresponding to $t=\infty$ is easily handled separately). Thus, to investigate the rôle of the integral in Theorem 2, we construct sets $F(N, \varepsilon, 1)$ as massive as possible.

THEOREM 4. *Let $f(r)$ be a decreasing positive function on* $(0<r<\infty)$, *and $f(0+)=+\infty$. Then for an appropriate sequence $N=(N_k)$, and $\varepsilon(k)\to 0$, the set $F=F(N, \varepsilon, 1)$ carries a measure σ whose modulus of continuity ω, defined by*

$$\omega(y) = \sup \sigma(B), \qquad \text{diam } B \leqq y,$$

has the property

$$\int_r^\infty \omega^2(y) y^{-2}\, dy \ll f(r), \qquad 0 < r < 1.$$

When $\mu=\sigma\times\sigma$, in the plane, we have for any vector Y,

$$\int_{|X|-Y\geqq r} |X - Y|^{-1}\mu(dX) \ll \int_r^\infty \omega^2(y) y^{-2}\, dy,$$

so this theorem completes Theorem 2.

The canonical product measure on the dyadic set F_0 of all sums $\sum \pm 2^{-m_j}$ has modulus of continuity $\omega(2^{-m_j}) \ll 2^{-j}$. Moreover, F_0 will be contained in a set $F(N, \varepsilon, 1)$ provided $\sup(m_{j+1}-2m_j)=+\infty$. To estimate the integral $\int_r^\infty$, we suppose that $r=2^{-m_s}$ and sum over the intervals $2^{-m_2} \leqq y \leqq 2^{-m_1}, \cdots$, obtaining

$$\int_r^\infty \omega^2(y) y^{-2}\, dy \ll \sum_1^s 4^{-m_j} 2^{m_j} \equiv \Phi(s), \quad \text{when } r = 2^{-m_s}.$$

Suppose now that $M=(m_j)$ is an increasing sequence of positive integers, that $m_{j+1}=m_j+1$ for $j \geqq j_1$ and $\Phi(s) \leqq cf(s^{-1})$ for $s \geqq 1$. We claim that if p is sufficiently large the sequence $\bar{M}$ defined by

$$\bar{m}_j = m_j, \quad 1 \leqq j < p, \qquad \bar{m}_j = m_j + m_p + k, \quad p \leqq j < \infty$$

has the same properties. In fact $\Phi(s)$ is unchanged when $s<p$, and otherwise $\Phi(s)$ is increased by

$$2^{m_p} 2^k \sum_p^s 4^{-j} 2^{m_j} \leqq 2^{k+1} 4^{m_p} 4^{-p}.$$

Using again the relation $m_{j+1}=m_j+1$ for $j \geqq j_1$ we can estimate this increment by B_k, so the inequality on p becomes

$$\Phi(s) + B_k \leqq cf(s^{-1}) \quad \text{for } s \geqq p.$$

Now $f(0+)=+\infty$ while $\Phi(s)$ remains bounded, so the inequality written above is satisfied for large p. In the sequence $\bar{M}$ we have $\bar{m}_p \geqq 2\bar{m}_{p-1}+k$. This process, repeated for $k=1, 2, 3, \cdots$, yields a sequence with the necessary gaps, while $\Phi(s) \leqq cf(s^{-1})$ for $s \geqq 1$, and hence $4^{-s}2^{m_s} \leqq cf(s^{-1})$. Thus if $f(t)<t^{-1}$ near 0 we have $m_s<3s$ for large s. To each small $r>0$ there is an s such that $r^4 \leqq 2^{-m_s}<r$, and

$$\int_r^\infty \omega^2(y) y^{-2}\, dy \ll \Phi(s) \leqq cf(s^{-1}) \leqq cf(r).$$

Although we have a satisfactory converse to Theorem 2, it remains to construct a set E of positive 1-measure or, if possible, non-σ-finite 1-measure, such that E carries no measure μ whose Fourier-Stieltjes transform $\hat{\mu}(r \cos\theta, r \sin\theta)$ tends to 0 along almost all lines. Unfortunately all the sets $F(N, \varepsilon, 1)$ have 1-measure 0.

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