EXTENDING A JORDAN RING HOMOMORPHISM

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Abstract. In this paper a homomorphism from an ideal \( \mathcal{B} \) of a quadratic Jordan algebra \( \mathcal{J} \) without 2-torsion over a ring \( \Phi \) onto a unital quadratic Jordan algebra \( \mathcal{J}' \) without 2-torsion is extended to a homomorphism from \( \mathcal{J} \) to \( \mathcal{J}' \). We then show if \( D \) is any class of quadratic Jordan algebras without 2-torsion, then the upper radical property determined by \( D \) is hereditary.

1. Preliminaries. We adopt the notation and terminology of an earlier paper [3] concerning quadratic Jordan algebras and hereditary radical properties. For a discussion of upper radical properties the reader is referred to [1] and [2]. Basically, given a class of rings \( D \) with the property that any nonzero ideal of a ring of \( D \) can be mapped homomorphically onto a ring of \( D \), then \( D \) can be extended to a class of all \( D \) semisimple rings: a ring is \( D \) radical if it cannot be mapped homomorphically onto a ring of \( D \); a ring without \( D \) radical ideals is \( D \) semisimple.

2. Two theorems.

Theorem 1. Given a quadratic Jordan algebra \( \mathcal{J} \) with ideal \( \mathcal{B} \) and a unital quadratic Jordan algebra \( \mathcal{J}' \) without 2-torsion, a homomorphism \( \varphi \) from \( \mathcal{B} \) onto \( \mathcal{J}' \) can be extended to a homomorphism \( \bar{\varphi} : \mathcal{J} \rightarrow \mathcal{J}' \).

Proof. Since \( \varphi \) is an onto homomorphism, some element of \( \mathcal{B} \), say \( b \), maps into 1. Define \( \bar{\varphi} : \mathcal{J} \rightarrow \mathcal{J}' \) by \( \bar{\varphi}(a) = \varphi(U_b a) \). \( \bar{\varphi} \) is clearly linear. For \( b' \in \mathcal{B} \),

\[
\bar{\varphi}(b') = \varphi(U_b b') = U_{\varphi(b)} \varphi(b') = U_1 \varphi(b') = \varphi(b').
\]

It is therefore established that \( \bar{\varphi} \) extends \( \varphi \). It now remains to show that \( \bar{\varphi} \) is indeed a homomorphism. Since \( 2U_x = V_x^2 - V_x^2 \), under the assumption that \( \mathcal{J}' \) has no 2-torsion, it is sufficient to show that \( \bar{\varphi} \) preserves the \( V \) operator, i.e. to show \( \bar{\varphi}(a \circ a') = \bar{\varphi}(a) \circ \bar{\varphi}(a') \) for \( a, a' \in \mathcal{J} \). For then \( \bar{\varphi}(2U_{x,y}) = 2U_{\varphi(x), \varphi(y)} \bar{\varphi}(y) \) and \( 2\{\varphi(U_{x,y}) - U_{\varphi(x), \varphi(y)} \bar{\varphi}(y)\} = 0 \) implies \( \bar{\varphi}(U_{x,y}) - U_{\varphi(x), \varphi(y)} \bar{\varphi}(y) = 0 \). First a simplification: it will be convenient to express \( U_b U_b V_{a a'} \) as

\[
U_b[(b \circ a) \circ (b \circ a')] - (U_b a) \circ (U_b a') - b \circ \{b a U_b a'\},
\]

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an identity which is easily verified in the case of special Jordan algebras and is therefore true for all Jordan algebras as a consequence of Macdonald’s theorem [4]. Now, since \( \varphi(b') = \varphi(b') = \varphi(U_b b') \) for \( b' \in S \),
\[
\varphi(a \circ a') = \varphi(U_b(a \circ a')) = \varphi(U_b U_b(a \circ a')) = \varphi(U_b[(b \circ a) \circ (b \circ a')]) - (U_b a) \circ (U_b a') - b \circ \{b a U_b a'\}).
\]
Consider these terms one at a time.
\[
\varphi(U_b[(b \circ a) \circ (b \circ a')]) = \varphi(U_b[(V_b a) \circ (V_b a')]) = \varphi((V_b a) \circ (V_b a'))
\]
\[
= \varphi(V_b a) \circ \varphi(V_b a') = \varphi(U_b V_b a') \circ \varphi(U_b V_b a') = \varphi(V_b U_b a') \circ \varphi(V_b U_b a')
\]
\[
= \psi(U_b a) \circ \psi(U_b a') = 2 \psi(a) \circ 2 \psi(a') = 4(\psi(a) \circ \psi(a')).
\]
Next,
\[
\varphi[(U_b a) \circ (U_b a')] = \varphi(U_b a) \circ \varphi(U_b a') = \psi(a) \circ \psi(a').
\]
Finally,
\[
\varphi(b \circ \{b a U_b a'\}) = \varphi(V_b V_b a U_b a') = \varphi(V_b U_b U_b a a')
\]
\[
= \varphi(U_b(U_{b a} a)(b \circ a') + U_{b U_{b a} a}(b \circ a) - 2(U_b a) \circ (U_b a'))
\]
(by Macdonald’s theorem)
\[
= \varphi(U_b(U_{b a} a)(b \circ a') + U_{b U_{b a} a}(b \circ a) - 2(\psi(a) \circ \psi(a'))
\]
\[
= \varphi(U_b a)(U_b a') + (U_b a') \circ (U_b U_b a')
\]
\[
+ \varphi(U_b[(U_b a) \circ (U_b a')]) + (U_b a) \circ (U_b U_b a') - 2(\psi(a) \circ \psi(a'))
\]
(again using Macdonald’s Theorem)
\[
= 2(\psi(a) \circ \psi(a')).
\]
Combining these results, \( \varphi(a \circ a') = \psi(a) \circ \psi(a') \). \( \varphi \) is therefore a homomorphism and the proof of the theorem is complete.

**Theorem 2.** If \( D \) is any class of quadratic Jordan rings with unit element over a ring \( \Phi \) and no 2-torsion, satisfying the condition that any nonzero ideal of a ring of \( D \) can be mapped homomorphically onto a ring of \( D \), then the upper radical property determined by \( D \) is hereditary.

**Proof.** Let \( \mathcal{J} \) be a \( D \) radical ring and let \( \mathcal{B} \) be a nonzero ideal of \( \mathcal{J} \). Assume \( \mathcal{B} \) is not a \( D \) radical ring. Then \( \mathcal{B} \) can be mapped homomorphically onto some ring \( \mathcal{G} \) in the class \( D \). By the above theorem \( \mathcal{J} \) is also homomorphic to \( \mathcal{G} \) which contradicts \( \mathcal{J} \) being \( D \) radical.

**References**


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