ON A PROBLEM OF F. RIESZ CONCERNING PROXIMITY STRUCTURES

W. J. THRON

Abstract. It is shown that every separated Lodato proximity is induced by the elementary proximity on a $T_1$ bicompletion of the original space.

A basic proximity structure $\Pi$ on a set $X$ is a relation on $\mathcal{P}(X)$ satisfying the following requirements

$\Pi_1: \Pi = \Pi^{-1},$
$\Pi_2: A \cup B \in \Pi(C) \iff A \in \Pi(C) \text{ or } B \in \Pi(C),$
$\Pi_3: A \cap B \neq \emptyset \Rightarrow A \in \Pi(B),$
$\Pi_4: \emptyset \notin \Pi(A) \forall A \in \mathcal{P}(X).$

Here $\Pi(A) = \{B: \langle A, B \rangle \in \Pi\}.$ Each proximity structure induces a closure operator on $X$ as follows: $c_\Pi(A) = \{x: [x] \in \Pi(A)\}.$ If for a proximity relation the additional condition

$\Pi_6: c_\Pi(A) \notin \Pi(B) \Rightarrow A \in \Pi(B)$

also holds then $\Pi$ is called a LO-proximity. A relation $\Pi$ is said to be separated if in it

$\Pi_6: [x] \in \Pi([y]) \iff x = y,$

is valid.

A grill on $X$ is a family $\mathcal{G} \subset \mathcal{P}(X)$ which satisfies:

$\mathcal{G}_1: A \supset B \in \mathcal{G} \Rightarrow A \in \mathcal{G},$
$\mathcal{G}_2: A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G} \text{ or } B \in \mathcal{G},$
$\mathcal{G}_3: \emptyset \notin \mathcal{G}.$

Grills were introduced by Choquet [1] in 1947. It is known (see for example [5, Lemma 5.7]) that every grill is the union of ultrafilters. It is easy to verify that the converse also holds. It is an immediate consequence of $\Pi_2$ and $\Pi_4$ that $\Pi(A)$ is a grill for all $A \in \mathcal{P}(X).$

Define $b(\Pi, \mathcal{G}) = \{B: c_\Pi(B) \in \mathcal{G}\}.$ One easily verifies that if $\mathcal{G}$ is a grill then $b(\Pi, \mathcal{G})$ is a grill, $b(\Pi, \mathcal{G}) \supset \mathcal{G},$ and $\mathcal{G}_1 \supset \mathcal{G}_2$ implies $b(\Pi, \mathcal{G}_1) \supset b(\Pi, \mathcal{G}_2).$

Received by the editors December 13, 1972.

AMS (MOS) subject classifications (1970). Primary 54E05; Secondary 54D35.

Key words and phrases. Proximity, LO-proximity, elementary proximity, bunch, $\Pi$-clan, grill.
A grill $\mathfrak{G}$ for which it is true that $A, B \in \mathfrak{G} \Rightarrow A \in \Pi(B)$ will be called a \textit{$\Pi$-clan}. A \textit{$\Pi$-clan} $\mathfrak{G}$ which satisfies the additional condition $b(\Pi, \mathfrak{G}) = \mathfrak{G}$ is called a \textit{bunch}.

Let $c$ be a Čech closure operator on $X$ then the relation $\Pi_0$ on $X$ defined by

$$A \in \Pi_0(B) \Leftrightarrow c(A) \cap c(B) \neq \emptyset$$

is a basic proximity. It is called the \textit{elementary proximity} associated with $c$. It is not in general true that $c = c_*$. However, if $\Pi$ is a separated \textit{LO}-proximity then $c_*$ is the closure operator for a $T_1$-topology and if $c$ is a Kuratowski closure operator which generates a $T_1$-topology then $c = c_*$. Let two proximity spaces $(X, \Pi)$ and $(Y, \Pi^*)$ and an injection $\varphi: X \rightarrow Y$ be given. Then $\Pi$ is said to be induced by $\Pi^*$ if

$$A \in \Pi(B) \Leftrightarrow \varphi(A) \in \Pi^*(\varphi(B)).$$

The problem of Riesz \cite{6}, referred to in the title, is the following: what types of proximity can be induced by elementary proximities on suitably constructed extension spaces of the original space? Riesz posed the problem in 1908, suggested a possible approach (using maximal $\Pi$-clans) but gave no answer. Clearly, the problem suggests that there may be a close relation between the proximities compatible with a given topological space and a certain class of topological extensions of the space. For \textit{EF}-proximities Smirnov \cite{7} in 1952 showed that they are induced by elementary proximities on $T_2$-bicompactifications of the underlying space. Improving on earlier work of Leader \cite{3} and Lodato \cite{4} Gagrat and Naimpally \cite{2} recently showed that every separated \textit{LO}-proximity which satisfies the additional condition:

\textbf{GN}: Given $A \in \Pi(B)$ there exists a bunch $\mathfrak{B}$ such that $A, B \in \mathfrak{B}$, is induced by the elementary proximity on a $T_1$-bicompactification of the original space.

We shall show that every \textit{LO}-proximity satisfies \textbf{GN} (Theorem 4) and hence every separated \textit{LO}-proximity can be induced by an elementary proximity. Harris has coined the name \textit{WI}-proximity for those proximities which can be induced by an elementary proximity. He has shown that every separated \textit{WI}-proximity is a \textit{LO}-proximity. It now follows that the separated \textit{WI}-proximities are exactly the separated \textit{LO}-proximities.

The result stated above is the final link in a chain whose other members are also of interest.

\textbf{Theorem 1}. \textit{Let $\mathfrak{F}$ be a filter and $\Pi$ a basic proximity on $X$; then $\Pi(\mathfrak{F}) = \bigcap \{\Pi(A) : A \in \mathfrak{F}\}$ is a grill.}
Proof. Clearly $\Pi(\bar{y})$ satisfies conditions $G_1$ and $G_3$. Now assume $A \cup B \in \Pi(\bar{y})$ and $A \notin \Pi(\bar{y})$, $B \notin \Pi(\bar{y})$. Then there exist sets $C$ and $D$ in $\bar{y}$ such that $A \notin \Pi(C)$, $B \notin \Pi(D)$. From this $A \notin \Pi(C \cap D)$, $B \notin \Pi(C \cap D)$ follows. Since $C \cap D \in \bar{y}$ we have $A \cup B \in \Pi(C \cap D)$ and thus a contradiction to the fact that $\Pi(C \cap D)$ is a grill.

Theorem 2. Let $\Pi$ be a basic proximity on $X$ then $A \in \Pi(B)$ implies the existence of a $\Pi$-clan $\mathcal{G}$ on $X$ such that $A, B \in \mathcal{G}$.

Proof. Since $\Pi(B)$ is a grill it is a union of ultrafilters. Hence there exists an ultrafilter $U_A$ such that $A \in U_A \subseteq \Pi(B)$. It follows from the symmetry of $\Pi$ that $B \in \Pi(U_A)$. Since $\Pi(U_A)$ is a grill it follows that there exists an ultrafilter $U_B$ such that $B \in U_B \subseteq \Pi(U_A)$. Since $U_B \subseteq \Pi(U_A)$ implies $U_A \subseteq \Pi(U_B)$ a desired $\Pi$-clan is $\mathcal{G} = U_A \cup U_B$.

Theorem 3. Let $\Pi$ be a LO-proximity on $X$ then every maximal $\Pi$-clan is a bunch with respect to $\Pi$.

Proof. If $\mathcal{G}$ is a $\Pi$-clan than $b(\Pi, \mathcal{G})$ is a $\Pi$-clan. To see this note that since $\Pi$ is a LO-proximity it satisfies $P_5$ and hence $b(\Pi, \Pi(A)) = \Pi(A)$. Since $\mathcal{G}$ is a $\Pi$-clan we have $\mathcal{G} \subseteq \Pi(A)$ for all $A \in \mathcal{G}$. Hence $b(\Pi, \mathcal{G}) \subseteq b(\Pi, \Pi(A)) = \Pi(A)$. By symmetry of $\Pi$, $\mathcal{G} \subseteq \Pi(B)$ for all $B \in b(\Pi, \mathcal{G})$ and hence $b(\Pi, \mathcal{G}) \subseteq \Pi(B)$ for all $B \in b(\Pi, \mathcal{G})$. It follows that $b(\Pi, \mathcal{G})$ is a $\Pi$-clan. For every maximal $\Pi$-clan $\mathcal{G}^*$ we then have $\mathcal{G}^* = b(\Pi, \mathcal{G}^*)$ (since $b(\Pi, \mathcal{G}) \supseteq \mathcal{G}$ for all grills $\mathcal{G}$). That is $\mathcal{G}^*$ is a bunch.

Theorem 4. Let $\Pi$ be a LO-proximity on $X$ and let $A \in \Pi(B)$. Then there exists a bunch $\mathcal{B}$ containing $A$ and $B$.

Proof. Let $\mathcal{G}$ be a $\Pi$-clan. There exists a maximal $\Pi$-clan $\mathcal{G}^*$ containing $\mathcal{G}$. This is proved using Zorn’s lemma. By Theorem 3 $\mathcal{G}^*$ is a bunch. By Theorem 2 a $\mathcal{G}$ can be found to contain $A$ and $B$, hence $\mathcal{G}^*$ contains the two sets.

A more extensive discussion of the ideas employed here is given in a forthcoming article [8] by the author.

References


Department of Mathematics, University of Colorado, Boulder, Colorado 80302