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THE EQUATION $L(E, X^{**}) = L(E, X)^{**}$ AND THE PRINCIPLE OF LOCAL REFLEXIVITY

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ABSTRACT. A new derivation of the equation $L(E, X^{**}) = L(E, X)^{**}$ is given, for $\dim(E) < \infty$ and X a Banach space. From this equation the principle of local reflexivity is derived.

0. **Introduction.** The principle of local reflexivity [6] in the somewhat stronger form found in [5] is derived here from the equation $L(E, X)^{**} = L(E, X^{**})$. This equation is found explicitly in Schatten's monograph [7, pp. 40, 41] and at least implicitly in [3, p. 13]. Thus it is a classic formula in the theory of tensor products of Banach spaces. In §3 we use nontensor product methods to derive the equation $L(E, X^{**}) = L(E, X)^{**}$. The argument is easily accessible to students in a first functional analysis course.

1. **Notation and preliminaries.** Always X, Y, Z are Banach spaces and A, E, F are finite dimensional Banach spaces. All operators S, T, U, V, W are continuous linear operators and the Banach space of operators from X to Y is denoted by $L(X, Y)$. Always X is identified with its natural embedding in X^{**}. If $T \in L(X, Y^{**})$ and $(T_\alpha) \subset L(X, Y)$ is a net such that $\lim f(T_\alpha x) = Tx(f)$ for each f in Y^*, x in X write w^*-op $\lim T_\alpha = T$ ($T_\alpha \to T$ in the weak-star operator topology).

LEMMA 1. *Let (T_α) be a net in $L(X, Y)$ and T in $L(X, Y^{**})$ with $\|T_\alpha\| \leqq \|T\|$ for each α. Suppose $A \subset X$, $TA \subset Y$ and w^*-op $\lim_\alpha T_\alpha = T$. Then, to $\varepsilon > 0$, there is a net $(S_\alpha) \subset L(X, Y)$ such that $\|S_\alpha\| < \|T\| + \varepsilon$, w^*-op $\lim S_\alpha = T$ and $S_\alpha a = Ta$ for each a in A.*

PROOF. For each a in A, $(T_\alpha a)$ converges weakly to Ta. Since $\dim(A) < \infty$, using standard techniques (e.g. [2, p. 477]), a net of convex combinations of (T_α), say (U_α), converges in norm on A, and w^*-lim $U_\alpha = T$. Write $X = A \oplus Z$ and set $S_\alpha(a+f) = Ta + U_\alpha f$. Then $S_\alpha \in L(X, Y)$ and $\|S_\alpha - U_\alpha\| \xrightarrow{\alpha} 0$. Thus, for large α, $\|S_\alpha\| < \|T\| + \varepsilon$.

If $\dim(E) = 1$, then $L(E, X)^{**} = L(E, X^{**})$ is simply the statement that the unit ball $U_1(X) = \{x, \|x\| \leqq 1\}$ is weak-star dense in $U_1(X^{**})$. As

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described below, the equation means that, for each T in $L(E, X^{**})$, there is a net $(T_\alpha) \subset L(E, X)$ such that $\|T_\alpha\| \leqq \|T\|$ and w^*-op lim $T_\alpha = T$. To see that this is the meaning of the equation let $e_1, \cdots, e_n$ be a basis for E and identify each T in $L(E, X)$ with the n-tuple (Te_i). In $Y = \prod_1^n X$ use the usual coordinatewise vector and scalar operations and set $\|(x_i)\| = \sup\{\|\sum a_i x_i\|, \|\sum a_i e_i\| \leqq 1\}$, so that the identification is an isometry between $L(E, X)$ and Y. Then $Y^* = \prod_1^n X^*$ with

$$\|(x_i^*)\| = \sup\left\{\sum x_i^*(x_i) \mid \|(x_i)\| < 1\right\}$$

and $Y^{**} = \prod_1^n X^{**}$ with $\|(x_i^{**})\| = \sup\{\sum x_i^{**}(x_i^*) | \ \|(x_i^*)\| \leqq 1\}$. Now associate each element (x_i^{**}) of $\prod_1^n X^{**}$ with the operator T such that $Te_i = x_i^{**}$. If $\|(x_i^{**})\| = 1$ and $\|(x_i^\alpha)\| \leqq 1$ such that $\sum x_i^*(x_i^\alpha) \to \sum x_i^{**}(x_i^*)$ for each (x_i^*) in Y^*, then $x^*(\sum b_i x_i^\alpha) \to (\sum b_i x_i^{**})(x^*)$ for each x^* in X^*. Thus $\lim \|\sum b_i x_i^\alpha\| \geqq \|\sum b_i x_i^{**}\|$. It easily follows that, if $\varepsilon > 0$, $\|(x_i^\alpha)\| = \sup\{\|\sum b_i x_i^\alpha\|, \|\sum b_i e_i\| \leqq 1\} \geqq (1-\varepsilon)\|T\|$ for large α or $\|(x_i^{**})\| \geqq \|T\|$.

In summary, the mapping $(x_i^{**}) \to T$ is a norm decreasing mapping from $L(E, X)^{**}$ onto $L(E, X^{**})$ which is continuous with the weak-star topology on $L(E, X)^{**}$ and the weak-star operator topology on $L(E, X^{**})$. Further it is the identity on $L(E, X)$. The equation $L(E, X^{**}) = L(E, X)^{**}$ means this mapping is an isometry.

2. **Local reflexivity.** Let F be subspace of X^* with basis $\{f_1, \cdots, f_k\}$ and let $T \in L(E, X^{**})$ with E having basis $\{e_1, \cdots, e_n\}$ such that $[e_1, \cdots, e_m] = E \cap X$. The pairs (f_i, e_j) define functionals on $L(E, X)$ by $(f_i, e_i)(S) = f_i(Se_i)$ (it is easy to compute that $\|(f_i, e_j)\| = \|f_i\| \ \|e_i\|$). Using Helly's theorem (e.g. [**8**, p. 103]), and the equation $L(E, X)^{**} = L(E, X^{**})$ there is an S such that $\|S\| < \|T\| + \varepsilon$ and $(f_i, e_j)(S) = Te_j(f_i)$ for each i, j. Thus $f(Se) = Te(f)$ for every f in F, e in E. (This argument is used in Lemma 1, [**4**].) One may assume, by enlarging F if necessary, that for each e there is a norm one f in F such that $(1-\varepsilon)\|Te\| < Te(f)$. Constructing $S = S_G$ for each $G \supset F$ such that $g(Se) = Te(g)$ for each g in G, e in E, and such that $\|S_G\| < \|T\|(1+\varepsilon)$, then w^*-op $\lim_G S_G = T$. By Lemma 1 there is a net $(T_\alpha) \subset L(E, X)$ such that w^*-op lim $T_\alpha = T$, $T_\alpha e_i = e_i$ if $i \leqq m$, $f(T_\alpha e) = Te(f)$ for each e in E, f in F, and $\|T_\alpha\| < \|T\|(1+2\varepsilon)$. Further, $(1-\varepsilon)\|Te\| < \|T_\alpha e\| < \|T\|(1+2\varepsilon)$ if $\|e\| \leqq 1$.

THEOREM 1 (LOCAL REFLEXIVITY). *Let $E \subset X^{**}$, $A = E \cap X$, and $F \subset X^*$. To $\delta > 0$, there is an S in $L(E, X)$ such that $(1-\delta)\|e\| < \|Se\| < (1+\delta)\|e\|$, $Sa = a$ for each a in A, and $f(Se) = e(f)$ for each e in E, f in F.*

PROOF. As in the calculation preceding Theorem 1, enlarging F if necessary, assume $(1-\delta/2)\|e\| < \sup\{e(f) | \ \|f\| \leqq 1, f \in F\}$. Letting T be the identity operator from E to X^{**} construct (T_α) such that $(1-\delta/2) <$

$\|T_\alpha e\|<(1+\delta)$ if $\|e\|=1$. Then $(1-\delta/2)\|e\|<\|T_\alpha e\|<\|e\|(1+\delta)$ for every e and set $S=T_\alpha$ for some α.

3. **The derivation of** $L(E, X^{**})=L(E, X)^{**}$. If $E=l_{1,n}$ then the derivation is as follows. Let $\{e_1, \cdots, e_n\}$ be the usual unit vector basis of $l_{1,n}=E$. For T in $L(E, X)$, $\|T\|=\sup\{\|\sum \alpha_i Te_i\|, \sum |\alpha_i|\leqq 1\}\leqq\max\{\|Te_i\|\}$. But $\|T\|\geqq\max\|Te_i\|$ since $\|e_i\|=1$ for each i. Thus $Y=\prod_1^n X$ has norm $\|(x_i)\|=\max\{\|x_i\|\}$. Then $Y^*=\prod_1^n X^*$ has norm, $\|(x_i^*)\|=\sum \|x_i^*\|$ and $Y^{**}=\prod X^{**}$ has norm, $\|(x_i^{**})\|=\max\{\|x_i^{**}\|\}$. The latter is the norm for $L(E, X^{**})$ so that the mapping of Y^{**} to $L(E, X^{**})$ in §1 is an isometry.

Now let E, $\varepsilon>0$ be given and let V be an operator on $l_{1,n}$ to E such that $V(\{u|\ \|u\|<1+\varepsilon\})\supset\{e|\ \|e\|\leqq 1\}$. That such $l_{1,n}$, V exist may be seen by embedding E^* into an $l_{\infty,k}$ in such a way that $\|e^*\|\geqq\|Ue^*\|\geqq(1-\alpha)\|e^*\|$ and choosing α small and $V=U^*$. If $T\in L(E, X^{**})$, then $TV\in L(l_{1,n}, X^{**})$ and $\|TV\|\leqq\|T\|$. Set $A=\{u|TVu\in X\}$. There is a net (S_α) in $L(l_{1,n}, X)$ such that $\|S_\alpha\|\leqq\|TV\|(1+\varepsilon)$, w^*-op lim $S_\alpha=TV$, and by Lemma 1 we find S_α such that $S_\alpha u=TVu$ if $u\in A$. In particular if $Vu=0$ then $S_\alpha u=0$. Define $T_\alpha\in L(E, X)$ by letting $T_\alpha e=S_\alpha u$ if $Vu=e$. Because $Vu=0$ implies $S_\alpha u=0$ one has that T_α is well defined and in $L(E, X)$. Moreover $T_\alpha V=S_\alpha$. If $\|e\|\leqq 1$ and $\|u\|<1+\varepsilon$ such that $Vu=e$, then $\|T_\alpha e\|=\|S_\alpha u\|\leqq\|S_\alpha\|(1+\varepsilon)\leqq \|TV\|(1+\varepsilon)^2\leqq\|T\|(1+\varepsilon)^2$. Finally $x^*(T_\alpha Vu)\to(TVu)x^*$ and so $x^*(T_\alpha e)\to (Te)(x^*)$ for every e in X^*. Thus w^*-op lim $T_\alpha=T$. Since $\varepsilon>0$ is arbitrary the mapping from $L(E, X)^{**}$ to $L(E, X^{**})$ at the end of §1 is an isometry. This concludes the derivation.

BIBLIOGRAPHY

1. W. J. Davis, *Remarks on finite rank projections*, J. Approximation Theory (to appear).

2. N. Dunford and J. T. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.

3. A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bull. Soc. Mat. São Paulo **8** (1956), 1–79.

4. W. B. Johnson, *On the existence of strongly series summable Markuschevich bases in Banach spaces*, Trans. Amer. Math. Soc. **157** (1971), 481–486. MR **43** #7914.

5. W. B. Johnson, H. P. Rosenthal, and M. Zippin, *On bases, finite dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math. **9** (1971), 488–506. MR **43** #6702.

6. J. Lindenstrauss and H. P. Rosenthal, *The L_p-spaces*, Israel J. Math. **7** (1969), 325–349. MR **42** #5012.

7. R. Schatten, *A theory of cross-spaces*, Ann. of Math. Studies, no. 26, Princeton Univ. Press, Princeton, N.J., 1950. MR **12**, 186.

8. A. Wilansky, *Functional analysis*, Blaisdell, Waltham, Mass., 1964. MR **30** #425.

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