A REMARK ON $C_\sigma$ SPACES

SIMEON REICH

Abstract. We give a simple new proof of the following result, conjectured by Effros and proved by Fakhoury: Let $E$ be a $C_\sigma$ space and $Z$ the set of extreme points of the unit ball of $E^*$. Then $Z \cup \{0\} = \{p \in E^*: \langle fg h, p \rangle = \langle f, p \rangle \langle g, p \rangle \langle h, p \rangle$ for all $f, g, h$ in $E\}$.

Let $C(X)$ be the Banach space of all continuous real valued functions on a compact Hausdorff space $X$, equipped with the supremum norm. If $\sigma: X \to X$ is an involutory homeomorphism, then $C_\sigma(X)$ is the subspace of $C(X)$ consisting of those $f$ in $C(X)$ which satisfy $f(\sigma x) = -f(x)$ for all $x$ in $X$. $e_x$ will stand for the point functional corresponding to a point $x$ in $X$. If $E$ is a Banach space, then we shall denote its unit ball by $B(E)$ and its conjugate space by $E^*$. The set of extreme points of a subset $Q$ of $E$ will be denoted by $\text{ext } Q$.

The following result was conjectured by Effros [2, Remark 8.4] and proved by Fakhoury [4, Theorem 15].

Theorem. If $E$ is a $C_\sigma$ space, then $\text{ext } B(E^*) \cup \{0\} = \{p \in E^*: \langle fg h, p \rangle = \langle f, p \rangle \langle g, p \rangle \langle h, p \rangle$ for all $f, g, h$ in $E\}$.

Fakhoury's proof is measure-theoretic in nature. Since this theorem appears to be useful (see, for instance, [4, Theorem 25] and [3, Theorem 11]), a simple different proof might perhaps be of some interest. The purpose of this note is to present such a proof. We shall need a few auxiliary propositions.

Lemma 1 [1, p. 89]. If $E = C_\sigma(X)$, then $\text{ext } B(E^*) = \{e_x: x \in X \text{ and } x \neq \sigma x\}$.

Lemma 2 [5, Proposition 3.5]. Let $E$ be a Banach space and $J^*$ the adjoint of the canonical map $J: E \to E^{**}$. If $y \in \text{ext } B(E^{**})$, then $J^*y$ belongs to the weak star closure of $\text{ext } B(E^*)$.

Since the dual of a $C_\sigma$ space is isometric to an $L$ space, its second dual is isometric to $C(Y)$ for some (extremally disconnected) compact Hausdorff space $Y$.

Received by the editors November 9, 1972.


Key words and phrases. $C_\sigma$ space, extreme point.
Lemma 3. Let $E = C_0(X)$ and $E^{**} = C(Y)$. Then $J(fgh) = J(f)J(g)J(h)$ for all $f, g, h$ in $E$.

Proof. Let $y$ belong to $Y$. The previous lemmas imply that $J^*e_y$ belongs to \{$e_x : x \in X$ and $x \neq ax \} \cup \{0\}$. Therefore $J(fgh)(y) = \langle J(fgh), e_y \rangle = \langle fgh, J^*e_y \rangle = \langle f, J^*e_y \rangle \langle g, J^*e_y \rangle \langle h, J^*e_y \rangle = J(f)(y)J(g)(y)J(h)(y)$.

Lemma 4. If $E = C(X)$, then the Theorem is true.

Proof. If $p \neq 0$, then the equality $\langle f, p \rangle = \langle f, p \rangle \langle 1, p \rangle^2$, valid for each $f$ in $E$, implies that either $\langle 1, p \rangle = 1$, or $\langle 1, p \rangle = -1$. Clearly we may assume that the former possibility holds. Then $p$ is multiplicative on $C(X)$, hence an extreme point of the positive face of $B(E^*)$.

Proof of the Theorem. By Lemma 3, $\langle p, J(f)J(g)J(h) \rangle = \langle p, J(fgh) \rangle = \langle fgh, p \rangle = \langle f, p \rangle \langle g, p \rangle \langle h, p \rangle = \langle p, J(f) \rangle \langle p, J(g) \rangle \langle p, J(h) \rangle$ for all $f, g, h$ in $E$. Let $K : E^* \to E^{**}$ be the canonical map. For each $r$ in $E^{**} = C(Y)$, the operator defined on $C(Y)$ by $s \mapsto rs$ is weak star continuous. Therefore $\langle p, rst \rangle = \langle p, r \rangle \langle p, s \rangle \langle p, t \rangle$ for all $r, s, t$ in $C(Y)$. By Lemma 4, $Kp$ belongs to $\text{ext } B(C(Y)^*) \cup \{0\}$. Since $p = J^*Kp$, an appeal to Lemmas 1 and 2 concludes the proof.

Acknowledgement. I wish to thank my friend, Micha Sharir, for his valuable help.

References


Department of Mathematics, The Technion–Israel Institute of Technology, Haifa, Israel

Department of Mathematical Sciences, Tel Aviv University, Tel Aviv, Israel